Economic Growth: Problem Set #3

Due Friday December 3

Please hand in *one of* Question 1 or Question 2, which will be graded. The rest will be reviewed in the recitation but should not be handed in.

1. **National debt in an OLG model.** Consider a standard two-period OLG model where there are $L_t > 0$ young workers at date $t = 0, 1, 2, \ldots$ each of whom is endowed with 1 unit of labor supplied inelastically and with preferences

$$U_t = \log c_t^1 + \beta \log c_{t+1}^2, \qquad 0 < \beta < 1$$

where c_t^a denotes their consumption at age $a = 1, 2$. Each worker maximizes utility subject to the budget constraints

$$0 \le c_t^1 \le w_t - s_t, \qquad \text{and} \qquad 0 \le c_{t+1}^2 \le R_{t+1} s_t$$

The number of young workers L_t grows at rate $n \ge 0$. There is a representative firm with production function

$$Y_t = A K_t^\alpha L_t^{1-\alpha}, \qquad 0 < \alpha < 1, \qquad A > 0$$

Physical capital fully depreciates each period, $\delta = 1$.

(a) Show that the dynamics of capital per worker $k_t \equiv K_t / L_t$ are given by

$$(1+n) k_{t+1} = \frac{\beta}{1+\beta}(1-\alpha) f(k_t), \qquad f(k_t) \equiv A k_t^\alpha$$

Show that there is a unique non-trivial steady state $k^* > 0$.

(b) Let k_{GR}^* denote the 'golden rule' level of capital per worker, solving

$$f'(k_{\text{GR}}^*) = 1 + n$$

Characterize the set of underlying parameters α, β etc such that $k^* < k_{\text{GR}}^*$, i.e., such that the economy is *dynamically efficient*. Does a higher value of α make it easier or harder to achieve dynamic efficiency? Explain.

Now suppose that there is a government that issues one-period bonds. Let B_{t+1} denote the aggregate quantity of bonds issued in period t maturing in period $t+1$. Each of these bonds pays gross real interest $R_{t+1} = 1 + r_{t+1}$. For simplicity, suppose there is no 'primary deficit' and that the government simply rolls-over its debt each period so that

$$B_{t+1} = R_t B_t$$

Young workers can now save by holding either physical capital or government bonds. Asset market clearing now requires

$$K_{t+1} + B_{t+1} = L_t s_t$$

(c) Let $b_t \equiv B_t/L_t$ denote bonds per worker. Show that the dynamics of capital and bonds per worker are given by a system of the form

$$k_{t+1} = g_1(k_t, b_t)$$
$$b_{t+1} = g_2(k_t, b_t)$$

Provide expressions for the functions $g_1(k, b)$ and $g_2(k, b)$.

(d) Construct a phase diagram for this system in (k_t, b_t) space. Partition the (k_t, b_t) space into regions where $k_{t+1} > k_t$ vs $k_{t+1} < k_t$ and where $b_{t+1} > b_t$ vs $b_{t+1} < b_t$. How if at all do thes dynamics depend on whether the economy is dynamically efficient or not?

(e) Let (k^*, b^*) denote steady-state levels of capital and bonds per worker. How many steady states does this system have? How if at all does this depend on whether the economy is dynamically efficient or not?

(f) Let $\boldsymbol{J}(k^*, b^*)$ denote the Jacobian of this system evaluated at (k^*, b^*). Characterize the local stability of each steady state. Characterize the global dynamics starting from any initial condition (k_0, b_0).

(g) Suppose initially parameters are such that $k^* > k^*_{\text{GR}}$. Now suppose the government can levy a lump-sum tax τ_t on young workers so that government debt evolves according to

$$B_{t+1} = R_t B_t - L_t \tau_t$$

What tax rate τ_t would enable the government to maintain a constant b^* in perpetuity? Explain how this could be used to implement an equilibrium converging to a steady state with $k^* = k^*_{\text{GR}}$.

2. **Borrowing constraints and heterogeneity in an OLG model.** Consider a two-period OLG model with a constant population normalized to size $L = 1$ for convenience. We now assume both young and old work. When young, workers are identical and supply 1 efficiency unit of labor each. When old, workers are heterogeneous. Worker $i \in [0, 1]$ supplies $h(i)$ efficiency units of labor with $h(i)$ drawn from a distribution $\Psi(h) = \text{Prob}[h(i) \leq h]$. Importantly, young workers know what $h(i)$ they will have when old. Since workers are otherwise identical, we will refer to h as a worker's type.

Workers have identical preferences and seek to maximize

$$U_t = \log c^1_t + \beta \log c^2_{t+1}, \qquad 0 < \beta < 1$$

subject to the their budget constraints, which differ depending on the their h in old age,

$$0 \leq c^1_t \leq w_t - s_t, \qquad \text{and} \qquad 0 \leq c^2_{t+1} \leq R_{t+1} s_t + w_{t+1} h$$

Notice that w_t is the wage per efficiency unit. For now, no other limit is imposed on their choice of savings s_t — so in particular some agents may want to borrow, choosing $s_t < 0$.

There is a representative firm with production function

$$Y_t = A K^\alpha_t H^{1-\alpha}_t, \qquad 0 < \alpha < 1, \qquad A > 0$$

Physical capital fully depreciates each period, $\delta = 1$. Since each young worker supplies 1 efficiency unit and old workers supply h units each and the population is constant, labor market clearing requires

$$H_t = 1 + \int h \, d\Psi(h) \equiv \bar{H}$$

where $\bar{H} > 0$ is the constant aggregate supply of efficiency units of labor.

(a) Show that the savings of a worker of type h are given by

$$s_t(h) = \frac{\beta}{1+\beta}\left(w_t - \frac{w_{t+1}}{\beta R_{t+1}}\, h\right)$$

Which types of workers want to borrow? Which types want to lend? Explain.

(b) Show that the dynamics of capital per effective worker $k_t \equiv K_t/\bar{H}$ have the form

$$k_{t+1} = z\, k_t^{\alpha}$$

for some constant coefficient $z > 0$ to be determined. [*Hint*: show that in equilibrium the ratio $x_{t+1} \equiv w_{t+1}/(R_{t+1}w_t)$ is constant].

Now suppose that workers may be *borrowing constrained*. In particular, suppose that it is difficult to obtain a loan secured by one's own future labor income and that as a consequence young workers must have savings

$$s_t \geq -\bar{b} w_t, \qquad \bar{b} > 0$$

where $\underline{b} > 0$ parameterizes the severity of the borrowing constraint [e.g., if $\bar{b} = 0$ then borrowing is prohibited, but if $\bar{b} = +\infty$ there is no borrowing constraint].

(c) Show that there is a cutoff type h_t^* such that all $h \geq h_t^*$ are borrowing constrained. How does h_t^* depend on the growth rate of wages w_{t+1}/w_t and the return on savings R_{t+1}? Give intuition for why workers with high h are more likely to be constrained.

(d) Show that the dynamics of capital per effective worker $k_t \equiv K_t/\bar{H}$ now have the form

$$k_{t+1} = z(\bar{b})\, k_t^{\alpha}$$

for some constant coefficient $z(\bar{b}) > 0$ to be determined. Is $z(\bar{b})$ increasing or decreasing in the borrowing limit $\bar{b}$? Give intuition for your result.

(e) Suppose we observed a range of countries $j = 1, 2, ...$ differing only in $\bar{b}_j$, explain how these countries would differ in terms of long-run levels of output per worker. Would these countries be predicted to differ in terms of the return on capital? If these countries were initially in autarky but then opened to international borrowing or lending, which countries would tend to be net lenders and which net borrowers?

3. **Endogenous growth with human capital accumulation.** Consider a growth model in continuous time. The planner seeks to maximize

$$\int_0^\infty e^{-\rho t}\left(\frac{C(t)^{1-\theta}-1}{1-\theta}\right)dt, \qquad \rho,\theta>0$$

subject to the production function

$$Y(t)=AK(t)^\alpha(u(t)H(t))^{1-\alpha}, \qquad 0<\alpha<1, \qquad A>0$$

and laws of motion for physical capital and human capital accumulation

$$\dot{K}(t)=Y(t)-C(t)-\delta K(t), \qquad 0<\delta<1$$

$$\dot{H}(t)=B(1-u(t))H(t)-\delta H(t), \qquad B>0$$

and given initial conditions $K(0)>0$ and $H(0)>0$. The interpretation here is that each worker has $H(t)$ units of human capital per unit time and the planner chooses what fraction of their time $u(t)\in[0,1]$ is spent producing goods $Y(t)$ with the remaining time $1-u(t)$ spent acquiring more human capital. To simplify the algebra, physical and human capital have a common depreciation rate δ and no physical capital is used to produce human capital.

(a) Setup up a current-value Hamiltonian and derive the key optimality conditions for $C(t)$, $K(t)$, $H(t)$ and $u(t)$.

(b) Let $c(t)\equiv C(t)/K(t)$ and $k(t)\equiv K(t)/H(t)$. Show that there is a unique balanced growth path where $C(t)$, $K(t)$, and $H(t)$ all grow at a constant rate g^* and where time spent producing goods $u(t)$ is a constant u^*. Solve for g^* and u^* and for the ratios c^* and k^* along the balanced growth path.

(c) Derive conditions on the parameters that are sufficient for $g^*>0$, $u^*\in[0,1]$ and for the transversality conditions to be satisfied.

(d) Suppose the initial condition $k(0)\equiv K(0)/H(0)$ differs from the value k^* along the balanced growth path in part (b). Characterize the transitional dynamics of $c(t)$, $k(t)$ and $u(t)$. Consider in particular the two cases $\alpha<\theta$ and $\alpha>\theta$.

[*Hint*: with some algebra you can reduce this to a linear differential equation in $x(t)\equiv(u(t)H(t)/K(t))^{1-\alpha}=(u(t)/k(t))^{1-\alpha}$]