MIT Economics

14.452
Fall 2021

Economic Growth: Problem Set #1

Due Friday November 5

Please only hand in Questions 3 and 4, which will be graded. The rest will be reviewed in the recitation but should not be handed in.

1. **Solow basics.** Consider a Solow model in continuous time with production function $y = k^\alpha$, constant savings rate s, depreciation rate δ, productivity growth g and labor force growth n.

   (a) Derive expressions for the elasticities of steady state capital and output with respect to the savings rate
   $$\frac{d \log k^*}{d \log s}, \qquad \frac{d \log y^*}{d \log s}$$
   How do these depend on the curvature of the production function α? Explain.

   (b) In cross-country data, investment rates I/Y range from about 0.05 to 0.50. Supposing $\alpha = 1/3$ and countries are otherwise identical, how much long-run cross-country variation in output per worker does that Solow model predict? Is this a large or small amount of variation relative to actual cross-country differences in output per worker?

   (c) Explain how an increase in α affects the levels of k^* and y^*. How if at all does your answer depend on the saving rate s? Does a larger α increase or decrease the model's ability to generate cross-country variation in output per worker, other things equal?

2. **Inada conditions.** Consider a Solow model in continuous time with production function $y = f(k)$ and for simplicity suppose no productivity or labor force growth, $g = n = 0$. We will assume $f'(k) > 0$ and $f''(k) \leq 0$ but not impose the usual Inada conditions.

   (a) Given a saving rate s and depreciation rate δ, derive conditions on $f'(0)$ and $f'(\infty)$ that ensure the existence of a unique interior steady state k^*.

   (b) Suppose the production function is linear, $y = Ak$ for some $A > 0$. Show that your conditions from part (a) cannot both be satisfied. What economic implications does this have?

3. **Growth and the elasticity of factor substitution.** Consider a Solow model with CES production function
   $$F(K, L) = \left[(A_K K)^{\frac{\sigma-1}{\sigma}} + (A_L L)^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}}, \qquad A_K, A_L > 0, \qquad \sigma > 0$$
   Again suppose for simplicity no productivity or labor force growth, $g = n = 0$.

(a) The *elasticity of substitution* between capital and labor is defined by

$$\varepsilon_{K,L} \equiv -\frac{d\log(K/L)}{d\log(R/w)}$$

Think of this as the proportional change in K/L along a given isoquant in response to a change in the factor price ratio R/w. Show that for a competitive firm, taking the rental rate R and wage rate w as given, the elasticity of substitution is $\varepsilon_{K,L} = \sigma$.

(b) Show that if $\sigma > 1$ this Solow economy is 'asymptotically Ak' in the sense that

$$\lim_{k\to\infty} \frac{f(k)}{k} = A$$

for some coefficient $A > 0$ to be determined. Does this guarantee sustained long run growth based on capital accumulation alone? Why or why not? Supposing factors are priced at their marginal products, what implications does this have for the long run labor share and long run capital share? Give economic intuition for the role played by the assumption $\sigma > 1$.

(c) Now suppose $\sigma < 1$ and suppose instead of constant A_K, A_L there is both *capital-augmenting* and *labor-augmenting* productivity growth

$$A_K(t) = e^{g_K t}A_K(0), \qquad \text{and} \qquad A_L(t) = e^{g_L t}A_L(0)$$

Moreover suppose capital-augmenting progress is *faster* than labor-augmenting progress

$$g_K > g_L > 0$$

Show that this economy converges to a balanced growth path where capital $K(t)$ and output $Y(t)$ grow at constant rate g and where the capital share is a constant, α_K. Solve for g and α_K along this balanced growth path. Explain why this capital-augmenting productivity is nonetheless consistent with balanced growth.

4. **Natural resource depletion in the Solow model.** Consider a Solow model where output is given by the Cobb-Douglas production function

$$Y(t) = K(t)^{\alpha}N(t)^{\beta}(A(t)L(t))^{1-\alpha-\beta}, \qquad 0 < \alpha, \beta < 1$$

where $N(t)$ denotes a stock of *natural resources* that exogenously depletes at rate $\phi > 0$

$$\dot{N}(t) = -\phi N(t)$$

The rest of the model is as standard with constant savings rate s, capital depreciation rate δ, productivity growth g and labor force growth n.

(a) Let g_Y, g_K denote the growth rates of output and the capital stock along a balanced growth path. Show that along any balanced growth path $g_Y = g_K$. Solve for g_Y.

Note: For the rest of this question assume parameters are such that $g_Y + \delta > 0$.

(b) Does the economy necessarily converge to a balanced growth path? Explain.

(c) Let $q(t)$ denote the marginal product of the natural resource. Characterize the dynamics of $q(t)$. Explain how $q(t)$ depends on the underlying growth rate of productivity g, labor force growth n, and rate of depletion ϕ. How if at all do your answers depend on the magnitudes of α and β? Explain.