Economic Growth

Lecture 9: Endogenous growth, part one

Chris Edmond

MIT 14.452

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Endogenous Growth

- Models so far have little to say about origins of sustained growth or determinants of cross-country productivity differences.

- We will soon turn to models where improvements in productivity are the result of deliberate investments in R&D-like activities.

- Requires departing from perfect competition.

- But sustained growth is still possible with perfect competition

  - Ak models

  - Ak models 'in disguise' [*linear accumulation* technology, e.g., Rebelo 1991]

  - Ak models 'in equilibrium' [*externalities* in physical or human capital accumulation, e.g., Romer 1986, Lucas 1988]

Outline

1. **One-sector Ak model**
   - Setup
   - Equilibrium dynamics
   - Policy effects

2. **Two-sector Ak model**
   - Setup
   - Balanced growth path

3. **Growth with externalities**
   - Externalities in capital accumulation
   - Planning problem

Outline

1. One-sector Ak model

Setup

Equilibrium dynamics

Policy effects

2. Two-sector Ak model

Setup

Balanced growth path

3. Growth with externalities

Externalities in capital accumulation

Planning problem

Representative Household

- Representative household of size $L(t)$ growing at rate n.

- Consumption per worker $c(t) \equiv C(t)/L(t)$.

- Balanced growth preferences, inelastic labor supply

$$U = \int_0^\infty e^{-\rho t} \left(\frac{c(t)^{1-\theta} - 1}{1-\theta} \right) e^{nt} \, dt$$

- Flow budget constraint in terms of net assets per worker

$$\dot{a}(t) = (r(t) - n)a(t) + w(t) - c(t)$$

- No-ponzi game constraint in terms of net assets per worker

$$\lim_{T \to \infty} \exp\Big(- \int_0^T (r(s) - n) \, ds \Big) a(T) \geq 0$$

Representative Household

- Consumption Euler equation

$$\frac{\dot{c}(t)}{c(t)} = \frac{r(t) - \rho}{\theta}$$

- Transversality condition

$$\lim_{T \to \infty} \exp\Big(- \int_0^T (r(s) - n)\, ds \Big) a(T) = 0$$

Representative Firm

- Production function in terms of capital per worker

$$y(t) = f(k(t)) = Ak(t), \qquad A > 0$$

- Factor prices

$$R(t) = A$$

$$w(t) = 0$$

- Hence real interest rate

$$r(t) = R(t) - \delta = A - \delta$$

- Remarks. Key is not the absence of diminishing returns, but rather that the Inada conditions are not satisfied. In particular, that $f'(\infty) = A > 0$.

  The two-sector Ak model we turn to next will restore the role of labor in production and so will have $w(t) > 0$.

Competitive Equilibrium

- A *competitive equilibrium* is a system of prices and quantities such that:

  (i) the representative household maximizes utility, taking prices as given

  (ii) the representative firm maximizes profits, taking prices as given

  (iii) markets clear, in particular
  $$a(t) = k(t)$$

- The competitive equilibrium here is efficient. Finding a competitive equilibrium equivalent to solving planning problem.

- Key equilibrium conditions reduce to
  $$\dot{k}(t) = (A - \delta - n)k(t) - c(t)$$
  $$\dot{c}(t) = \frac{A - \delta - \rho}{\theta} c(t)$$
  plus the transversality condition
  $$\lim_{T \to \infty} \exp\Big(- (A - \delta - n)T \Big) k(T) = 0$$

Outline

Equilibrium Dynamics

- To simplify notation, let $g \equiv (A - \delta - \rho)/\theta$ and let $r \equiv A - \delta$.

- System can be written

$$\dot{k}(t) = (r - n)k(t) - c(t), \qquad \dot{c}(t) = gc(t)$$

- Growth in consumption per worker is constant g, independent of the level of capital per worker.

- Growth in capital per worker is constant if $c(t)/k(t)$ ratio is constant, in which case $k(t)$ must also be growing at rate g.

- Let's verify this is indeed the solution by solving the dynamics exactly.

Equilibrium Dynamics

- System can be written

$$\dot{k}(t) = (r-n)k(t) - c(t), \qquad \dot{c}(t) = gc(t)$$

- Implies a *second-order differential* equation in $k(t)$, namely

$$\ddot{k}(t) = (r-n+g)\dot{k}(t) - (r-n)gk(t)$$

- Look for solutions of the form $k(t) = e^{\lambda t}k(0)$ for some λ to be determined.

- Roots λ must satisfy the characteristic polynomial

$$\lambda^2 - (r-n+g)\lambda + (r-n)g = 0$$

so roots are

$$\lambda_1 = r-n, \qquad \lambda_2 = g$$

- Both may be positive. How do we know which root to pick?

Equilibrium Dynamics

- Suppose we take $\lambda_1 = r - n$. Then $\dot{k}(t) = (r-n)k(t)$ and from the resource constraint we would have

$$c(t) = (r-n)k(t) - \dot{k}(t) = 0$$

- We can do better than this [also violates the transversality condition].

- Suggests we should choose $\lambda_2 = g$. Then $\dot{k}(t) = gk(t)$ and from the resource constraint we would have

$$c(t) = (r-n)k(t) - \dot{k}(t) = (r-n-g)k(t)$$

- To satisfy the transversality condition we need

$$\lim_{t\to\infty} e^{-(r-n)t}e^{gt}k(0) = 0 \qquad \Leftrightarrow \qquad r-n > g$$

- Remark. So here transversality condition is satisfied iff the policy function $c = (r-n-g)k$ is increasing in k.

Equilibrium Dynamics

- Assumptions. To streamline the exposition, suppose

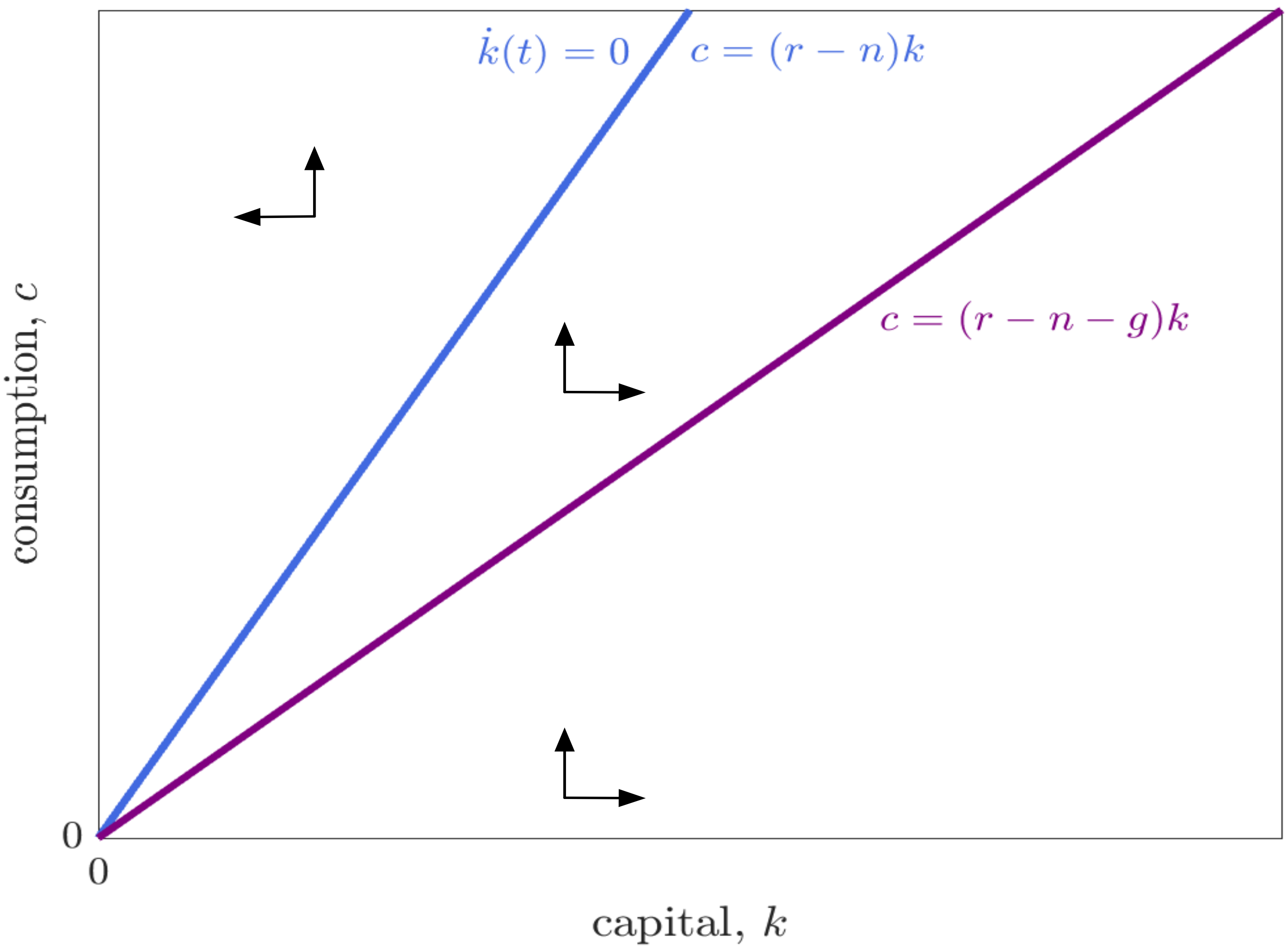
  (i) $A > \rho + \delta$, so economy is growing, $g > 0$
  (ii) $\rho > (1-\theta)(A-\delta) + \theta n$, so TVC is satisfied, $r - n > g$

- Then we can say
$$\frac{\dot{c}(t)}{c(t)} = \frac{\dot{k}(t)}{k(t)} = g > 0$$
with initial consumption given by policy function $c(0) = (r - n - g)k(0)$.

- No transitional dynamics, economy immediately on balanced growth path.

- Growth is sustained and endogenous
$$g = \frac{A - \delta - \rho}{\theta}$$
Higher ρ reduces the growth rate, higher intertemporal elasticity of substitution $1/\theta$ increases the growth rate, etc.

Phase Diagram



Saving Rate

- Saving rate is given by

$$s(t) \equiv \frac{I(t)}{Y(t)} = \frac{\dot{K}(t) + \delta K(t)}{Y(t)} = \left(\frac{\dot{K}(t)}{K(t)} + \delta\right)\frac{K(t)}{Y(t)} = \frac{g + n + \delta}{A}$$

so on substituting for g

$$s(t) = \frac{A - \delta - \rho + \theta(n + \delta)}{A\theta} \equiv s \in (0, 1), \qquad \text{for all } t$$

- Constant saving rate s, as in basic Solow model, but now endogenous to structural parameters $A, \delta, \rho, \theta, n$.

- Remark. (i) $g > 0$ ensures $s > 0$, (ii) $r - n > g$ ensures $s < 1$.

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Policy Effects

- Competitive equilibrium is Pareto efficient.

- To illustrate policy effects, consider distortionary tax $\tau \in (0,1)$ on return on net assets so that

$$\dot{a}(t) = ((1-\tau)r(t) - n)a(t) + w(t) - c(t)$$

- From consumption Euler equation, growth rate is now

$$g = \frac{(1-\tau)(A-\delta) - \rho}{\theta}$$

and saving rate is now

$$s = \frac{(1-\tau)(A-\delta) - \rho + \theta(n+\delta)}{A\theta}$$

- Both growth rate g and saving rate s are decreasing in τ whenever $A > \delta$.

Policy Effects

- Saving rate is now endogenous to policy τ.
- Since saving rate is constant, differences in policy τ will lead to *permanent* differences not just in levels but also in growth rates.
- Hence even smaller differences in τ can have very large effects.
- Consider two economies identical except one has tax τ and the other τ'.
- For any $\tau' > \tau$

$$\lim_{t\to\infty} \frac{Y(t\,;\tau')}{Y(t\,;\tau)} = 0$$

Policy Effects: Discussion

- So in principle Ak model can generate arbitrarily large differences in output per worker across countries.

- Given that, why not make Ak model the benchmark, rather than standard neoclassical model? Two key problems:

  (i) Ak model implies capital share of income $\rightarrow 1$, no labor demand

  (ii) Ak model implies *too much instability* in the cross-country distribution of income, predicts large amounts of churn based on policy variation — as opposed to relatively stable distribution we observe

  in other words, Ak model implies *too much sensitivity to policy*

- As we will see, the former problem is relatively straightforward to address.

- The latter problem is more fundamental.

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Two-Sector Ak Model

- Key implications of Ak model *not* driven by output linear in capital.

- Instead, key is that the *accumulation technology* is linear.

- Rebelo (1991) is a clean example, consumption good sector and investment good sector with linear accumulation of investment goods.

- Representative household with balanced growth preferences.

- Constant labor force $L > 0$ with $n = 0$ [this is not important].

Outline

Two-Sector Ak Model

- **Consumption Goods Sector.** Uses the production function

$$C(t) = BK_C(t)^\alpha L_C(t)^{1-\alpha}, \qquad 0 < \alpha < 1, \qquad B > 0$$

- **Investment Goods Sector.** Uses the production function

$$I(t) = AK_I(t), \qquad A > 0$$

- Investment goods sector more capital intensive than rest of the economy.

- Capital accumulation is then

$$\dot{K}(t) = I(t) - \delta K(t)$$

Market Clearing

- Labor is only used to make consumption goods

$$L_C(t) = L$$

- Capital is used in both sectors

$$K_C(t) + K_I(t) = K(t)$$

- Equilibrium has to pin down allocation of capital between sectors.

- Let $x(t) \equiv K_I(t)/K(t)$ denote *share* of capital used to make investment.

- Let $p_I(t)$, $p_C(t)$ denote prices of investment and consumption goods (we will make consumption goods the numeraire, but not yet).

- Relative price will adjust endogenously.

Consumption Euler Equation

- Let $r_C(t)$ denote the real return in *units of consumption goods.*

- Let $r_I(t)$ denote the real return in *units of investment goods.*

- These returns are linked by no-arbitrage arguments

$$\frac{r_C(t)}{p_C(t)} + \frac{\dot{p}_C(t)}{p_C(t)} = \frac{r_I(t)}{p_I(t)} + \frac{\dot{p}_I(t)}{p_I(t)}$$

- As in one-sector model, $r_C(t)$ satisfies consumption Euler equation

$$g_C(t) \equiv \frac{\dot{c}(t)}{c(t)} = \frac{r_C(t) - \rho}{\theta}$$

- As in one-sector model, $r_I(t)/p_I(t) = A - \delta$, constant.

- But what determines change in relative prices linking these returns?

Allocation of Capital Between Sectors

- Marginal revenue product of capital in investment sector

$$p_I(t)A$$

- Marginal revenue product of capital in consumption sector

$$p_C(t)\alpha B\Big(\underbrace{(1-x(t))K(t)}_{=K_C(t)}\Big)^{\alpha-1}L^{1-\alpha}$$

- So, in equilibrium, relative price will satisfy

$$\frac{p_I(t)}{p_C(t)} = \big((1-x(t))K(t)\big)^{\alpha-1} \times \left(\frac{\alpha B L^{1-\alpha}}{A}\right)$$

- Share of capital $x(t)$ in investment sector increasing in $p_I(t)/p_C(t)$.

Labor Demand and Wages

- Labor is used in the consumption sector.

- Perfect competition in goods and factor markets so

$$w(t) = p_C(t)(1-\alpha)B\big((1-x(t))K(t)\big)^{\alpha}L^{-\alpha}$$

- A competitive equilibrium is then defined in the usual way.

- As in basic Ak model, there are no transitional dynamics.

- Capital used in two sectors, but only one *type* of capital, frictionlessly allocated between sectors $\Rightarrow$ can jump straight to balanced growth path.

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Balanced Growth Path

- Let consumption goods be the numeraire, $p_C(t) = 1$.
- Look for *balanced growth path* with constant growth rates g_C, g_K etc and constant allocation x between sectors.
- Then from relative price condition, given x constant

$$\frac{\dot{p}_I(t)}{p_I(t)} = -(1-\alpha)g_K$$

- Hence from no-arbitrage condition

$$r_C(t) = \frac{r_I(t)}{p_I(t)} + \frac{\dot{p}_I(t)}{p_I(t)} = A - \delta - (1-\alpha)g_K$$

- Hence from consumption Euler equation

$$g_C = \frac{A - \delta - (1-\alpha)g_K - \rho}{\theta}$$

Balanced Growth Path

- Recall production function for consumption goods

$$C(t) = BK_C(t)^{\alpha}L_C(t)^{1-\alpha}, \qquad 0 < \alpha < 1$$

- So given that $L_C(t) = L$ constant and $K_C(t) = (1-x)K(t)$ we have

$$g_C = \alpha g_K < g_K$$

- Hence solution is

$$g_K = \frac{A - \delta - \rho}{1 - \alpha + \alpha\theta}, \qquad g_C = \alpha\,\frac{A - \delta - \rho}{1 - \alpha + \alpha\theta}$$

- From the labor market condition, wage growth is given by

$$g_W = \alpha g_K = g_C$$

- From the production function for investment goods

$$g_I = g_K$$

Allocation of Capital Between Sectors

- What pins down share x of capital in the investment goods sector?

- From accumulation technology

$$\frac{\dot{K}(t)}{K(t)} = \frac{I(t)}{K(t)} - \delta = Ax - \delta$$

- Hence along a balanced growth path

$$x = \frac{g_K + \delta}{A} = \frac{A - \rho - \alpha(1-\theta)\delta}{A(1 - \alpha(1-\theta))}$$

- Assumptions*. Now we need

(i) $A > \rho + \delta$, so economy is growing, $g_K, g_C > 0$ [same as in one-sector model]
(ii) $\rho > (1-\theta)\alpha(A - \delta)$, so TVC is satisfied [and $x < 1$]

Aggregate Expenditure

- Real aggregate expenditure/income in units of consumption

$$Y(t) = C(t) + p_I(t)I(t)$$

or

$$\frac{\dot{Y}(t)}{Y(t)} = \frac{C(t)}{Y(t)}\frac{\dot{C}(t)}{C(t)} + \frac{p_I(t)I(t)}{Y(t)}\left(\frac{\dot{p}_I(t)}{p_I(t)} + \frac{\dot{I}(t)}{I(t)}\right)$$

- So along a balanced growth path

$$g_Y = \frac{C(t)}{Y(t)}g_C + \frac{p_I(t)I(t)}{Y(t)}(-(1-\alpha)g_K + g_I)$$

$$= \frac{C(t)}{Y(t)}\alpha g_K + \left(1 - \frac{C(t)}{Y(t)}\right)\alpha g_K$$

$$= \alpha g_K = g_C$$

Discussion

- Unlike standard neoclassical growth model, there is continuous *capital deepening*, $K(t)/L$ grows without bound.

- Diminishing returns to capital in production, but linear accumulation.

- Physical capital $K(t)$ grows faster than $Y(t)$, which may seem in tension with constant 'capital/output' ratio.

- But in two-sector model $K(t)$ and $Y(t)$ have *different units.*

- Relative price of investment falls at just the right rate to keep expenditure share $p_I(t)I(t)/Y(t)$ constant.

Outline

Growth With Externalities

- Romer (1986) model of 'knowledge accumulation'.

- How to do knowledge accumulation in a competitive economy?

- Key idea: Marshallian *external economies*

  – knowledge accumulation *unintentional byproduct* of capital accumulation

  – a form of technological spillovers, simply 'in the air'

  – captures key idea that knowledge is largely *non-rival*

  – in short, positive externalities in capital accumulation

- Do this in one-sector model. Capital and output measured in same units.

Setup

- Constant labor force $L > 0$ with $n = 0$ [but now this is important].

- Continuum $i \in [0, 1]$ of identical firms each with production function

$$Y_i(t) = F(K_i(t), A(t)L_i(t))$$

- Factor market clearing

$$\int_0^1 K_i(t)\, di = K(t)$$

$$\int_0^1 L_i(t)\, di = L$$

- All firms have productivity $A(t)$. Hence all firms choose *the same* capital/labor ratio $K_i(t)/L_i(t) = K(t)/L(t)$ with factor prices

$$R(t) = F_K(K_i(t), A(t)L_i(t)) = F_K(K(t), A(t)L)$$

$$w(t) = F_L(K_i(t), A(t)L_i(t))A(t) = F_L(K(t), A(t)L)A(t)$$

Outline

Externalities in Capital Accumulation

- **Key assumption.** Each individual firm takes $A(t)$ as given but $A(t)$ is *endogenous* to aggregate $K(t)$.

- Economy-wide capital accumulation generates $A(t)$ as unintentional byproduct. No resources used, no payments to knowledge per se.

- Extreme version of such externalities

$$A(t) = BK(t), \qquad B > 0$$

- Implies aggregate technology is linear in aggregate capital

$$Y(t) = F(K(t), BK(t)L) = f(L)K(t), \qquad f(L) \equiv F(1, BL)$$

- REMARKS. A bit like 'learning-by-doing' but at the aggregate level. Lucas (1988) is similar, but externalities though human capital.

Externalities in Capital Accumulation

- A *competitive equilibrium* is a system of prices and quantities such that:

  (i) the representative household maximizes utility, taking prices as given

  (ii) the representative firm maximizes profits, taking prices as given

  (iii) markets clear, in particular
  $$a(t) = k(t)$$

- The competitive equilibrium here is generally *inefficient*, benefits from knowledge accumulation are not *internalized.*

- Factor prices satisfy

$$R(t) = F_K(K(t), A(t)L) = F_K(1, BL)$$

$$w(t) = F_L(K(t), A(t)L)A(t) = F_L(1, BL)BK(t)$$

Discussion

- Aggregate technology

$$Y(t) = f(L)K(t)$$

- But in equilibrium factor prices are

$$R(t) = F_K(1, BL) = [f(L) - f'(L)L] \leq f(L)$$

and

$$w(t) = F_L(1, BL)BK(t) = f'(L)K(t)$$

- Externality from capital accumulation to productivity is '*not priced*'.

- Rental rate constant $R = f(L) - f'(L)L$, real interest rate $r = R - \delta$.

Equilibrium Growth

- From consumption Euler equation, growth is

$$\frac{\dot{c}(t)}{c(t)} = \frac{r-\rho}{\theta} = \frac{f(L) - f'(L)L - \delta - \rho}{\theta} \equiv g$$

- From flow budget constraint, evaluated at equilibrium factor prices

$$\dot{k}(t) = rk(t) + w(t) - c(t) = (\underbrace{f(L) - \delta}_{\geq r})k(t) - c(t)$$

- Solution is again given by a constant policy function

$$c(t) = (f(L) - \delta - g)k(t)$$

- Assumptions*. Now we need

(i) $f(L) - f'(L)L > \rho + \delta$, so economy is growing, $g > 0$
(ii) $\rho > (1-\theta)(f(L) - f'(L)L - \delta)$, so TVC is satisfied, $r > g$ [note $n = 0$]

- Remark. If $r > g$ then policy function is increasing in k.

Scale Effects Problem

- Need $n = 0$ because of the *scale effects problem*. Write growth

$$g = \frac{f(L) - f'(L)L - \delta - \rho}{\theta}$$

- Implies larger economies, as measured by L, grow faster $[f''(L) < 0]$.

- With $n > 0$, growth rate g increases as L increases, perpetual acceleration.

- Capital accumulation so rapid transversality condition cannot be satisfied.

- Partly a consequence of strong form of capital externalities, we will see how to mitigate this next class.

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Planning Problem

- Given externality, not surprising competitive equilibrium is inefficient.

- What then is the efficient allocation? Planner maximizes

$$U = \int_0^\infty e^{-\rho t} \left(\frac{c(t)^{1-\theta} - 1}{1-\theta} \right) dt$$

subject to the flow resource constraint, *internalizing knowledge spillovers*

$$\dot{k}(t) = (f(L) - \delta)k(t) - c(t)$$

- Incentives to accumulate capital driven by the *social return* $f(L) - \delta$ which is larger than the *private return* $r = f(L) - f'(L)L - \delta$.

Planning Problem

- From Planner's consumption Euler equation

$$g^* = \frac{f(L) - \delta - \rho}{\theta} > \frac{f(L) - f'(L)L - \delta - \rho}{\theta} = g$$

- Planner internalizes effect of capital accumulation on productivity, so accumulates capital at faster rate than in decentralized equilibrium.

- Acknowledging higher social return to capital, planner chooses higher saving rate, trading off lower initial consumption for higher lifetime utility.

Summary

- Linearity, mostly clearly visible in Ak model, drives results
  - makes sustained growth possible, by violating Inada conditions
  - simplifies solution, eliminating transitional dynamics
- Important tension
  - standard parameterizations of neoclassical growth models make it *hard* to generate large cross-country differences in output per worker
  - linear growth models make it *too easy* to to generate large cross-country differences in output per worker
  - implicit 'each country is an island' interpretation seems particularly unattractive when thinking about knowledge spillovers

Next Class

- Endogenous growth as a consequence of deliberate investment in R&D.

- Further implications of economies of scale.

- Departing from perfect competition.

Homework

- Consider the decentralized equilibrium of the Romer (1986) model with production function

$$Y = K^{\alpha}(AL)^{1-\alpha}$$

- Now suppose that capital accumulation is *subsidized* at a constant rate τ funded by lump-sum taxes.

- Check. Show that the subsidy rate τ that implements the planner's solution is given by

$$1 + \tau = \frac{1}{\alpha}$$

- Check your intuition. Why is the subsidy rate decreasing in α?