# Economic Growth

## Lecture 8: Overlapping generations

Chris Edmond

MIT 14.452

Fall 2021

# Overlapping Generations

- Neoclassical growth model has all households present in all of infinitely-many periods.

- Overlapping generations model features perpetual arrival of *new generations* not present in earlier periods.

  [Allais (1947), Samuelson (1958), version here follows Diamond (1965)]

- Cross-sectional heterogeneity in age plays a fundamental role.

- Decisions made by older generations determine the opportunities facing younger generations.

# Overlapping Generations

- Influential framework, for two distinct reasons

  **(i)** *applied influence*: especially when lifecycle or demographics are crucial e.g., pensions/social security, health, fertility, etc

  **(ii)** *theoretical influence*: the 'double infinity' of commodities and agents leads to the possibility of Pareto inefficient competitive equilibria, i.e., first welfare theorem may not hold

  equilibria may be dynamically inefficient, capital overaccumulation

- We will begin with a simple illustration of the 'double infinity' issue in an exchange economy, then spend most of the lecture developing an OLG counterpart to the neoclassical growth model.

# Outline

1. **Notes on the economics of infinity**

2. **Benchmark two-period OLG growth model**
   - Setup
   - Savings function
   - Equilibrium uniqueness
   - Examples

3. **The possibility of dynamic inefficiency**

4. **Social security and capital accumulation**

# Setup

- Example follows Shell (1971).
- Exchange economy with infinitely-many dated commodities $t = 0, 1, 2, \ldots$
- Infinitely-many two-period lived individuals with linear utility

$$U_t = c_t^1 + c_{t+1}^2$$

  where $c_t^a$ denotes consumption on date $t$ of individual of age $a = 1, 2$.

- Individual $t$ endowed with 1 unit of date-$t$ commodity, budget set

$$p_t c_t^1 + p_{t+1} c_{t+1}^2 \leq p_t$$

- Allocation $\boldsymbol{c} = \{\boldsymbol{c}_t\}$, individual allocation $\boldsymbol{c}_t$ with elements $c_t^1, c_{t+1}^2 \geq 0$.
- Prices $\boldsymbol{p}$ with typical element $p_t \geq 0$. Let $p_0 = 1$ be the numeraire.

# No Trade is a Competitive Equilibrium

- A *competitive equilibrium* is a feasible allocation $\boldsymbol{c}$ and prices $\boldsymbol{p}$ such that

  **(i)** taking $\boldsymbol{p}$ as given, $\boldsymbol{c}_t$ is optimal for each individual $t$

  **(ii)** markets clear

$$c_t^1 + c_t^2 = 1$$

- Proposition. There is a competitive equilibrium with *no trade*, $c_t^1 = 1$ for all $t$, supported by prices $p_t = 1$ for all $t$.

- Proof. At $p_t = 1$, the budget set of individual $t$ is

$$c_t^1 + c_{t+1}^2 \leq 1$$

  Since $U_t = c_t^1 + c_{t+1}^2$, consumption $c_t^1 = 1$ is optimal for individual $t$. Hence $p_t = 1$ and $c_t^1 = 1$ for all $t$ is a competitive equilibrium.

- Remark. Here every individual consumes when young.

# No Trade is Not Pareto Efficient

- PROPOSITION. No trade competitive equilibrium is not Pareto efficient.

- PROOF (Sketch). Consider the following alternative

  - individual $t = 0$ consumes own endowment when young *and* individual $t = 1$ endowment when old

  - individual $t = 1$ consumes individual $t = 2$ endowment when old

  - individual $t = 2$ consumes individual $t = 3$ endowment when old

    $\vdots$

  - individual $t$ consumes individual $t + 1$ endowment when old

    $\vdots$

- Individual $t = 0$ is strictly better off and no other individual is worse off.

- In other words, *the first welfare theorem does not hold.*

# Discussion

- At these prices, market value of the aggregate endowment is infinite.
- Let $\boldsymbol{y}$ denote aggregate endowment of dated commodities.
- Since each individual $t$ has one unit of date-$t$ commodity, the aggregate endowment is the infinite stream

$$\boldsymbol{y} = (1, 1, 1, \dots)$$

- At these prices, the market value of the aggregate endowment is

$$\boldsymbol{p} \cdot \boldsymbol{y} = \sum_{t=0}^{\infty} p_t y_t = \sum_{t=0}^{\infty} 1 = +\infty$$

- So in attempting to apply the first welfare theorem we would not be able to conclude that Pareto dominant allocations must be budget infeasible [see Lecture 7 pages 14-15].
- Pareto inefficiency illustrated here gives rise to possibility of dynamic inefficiency in OLG growth model.

# Second Welfare Theorem

- Second welfare theorem *does not* require $\boldsymbol{p} \cdot \boldsymbol{y} < \infty$.

- Requires convex preferences and the cheaper point property, satisfied here.

- Can *implement* allocation where individual $t = 0$ consumes when young and old and every other individual consumes only when old.

- For each $t = 1, 2, \ldots$ tax individual $t$ one unit of date-$t$ commodity (when they are young) and give that unit to individual $t - 1$ (when they are old).

- This is a *within-period* tax/transfer between young and old at date $t$.

- Check. Allocation $\boldsymbol{c}_0 = (c_0^1, c_1^2) = (1, 1)$ and $\boldsymbol{c}_t = (c_t^1, c_{t+1}^2) = (0, 1)$ for all $t = 1, 2, \ldots$ with prices $p_t = 1$ for all $t$ is a competitive equilibrium.

- Example here is very special, but we will see similar mechanism can work more generally in OLG economies.

# Outline

1. **Notes on the economics of infinity**

2. **Benchmark two-period OLG growth model**
   - Setup
   - Savings function
   - Equilibrium uniqueness
   - Examples

3. **The possibility of dynamic inefficiency**

4. **Social security and capital accumulation**

# Outline

1. **Notes on the economics of infinity**

2. **Benchmark two-period OLG growth model**
   - Setup
   - Savings function
   - Equilibrium uniqueness
   - Examples

3. **The possibility of dynamic inefficiency**

4. **Social security and capital accumulation**

# Benchmark Two-Period OLG Model

- Discrete time $t = 0, 1, 2, \ldots$
- Individual born at $t$ lives $t$ and $t + 1$.
- Separable utility

$$U_t = u(c_t^1) + \beta u(c_{t+1}^2), \qquad 0 < \beta < 1$$

  where $c_t^a$ denotes consumption on date $t$ of individual of age $a = 1, 2$.

- To streamline exposition, assume $u'(c) > 0$, $u''(c) < 0$ and $u'(0) = +\infty$.
- Individuals inelastically supply one unit of labor when young.
- But need to save for when they are old.

# Individual Consumption/Savings Problem

- Individual born at date $t$ chooses consumption and savings $s_t$ to maximize

$$U_t = u(c_t^1) + \beta u(c_{t+1}^2), \qquad 0 < \beta < 1$$

  subject to

$$c_t^1 + s_t \leq w_t$$

  and

$$c_{t+1}^2 \leq R_{t+1} s_t$$

  taking wage rate $w_t$ and gross return on saving $R_{t+1}$ as given.

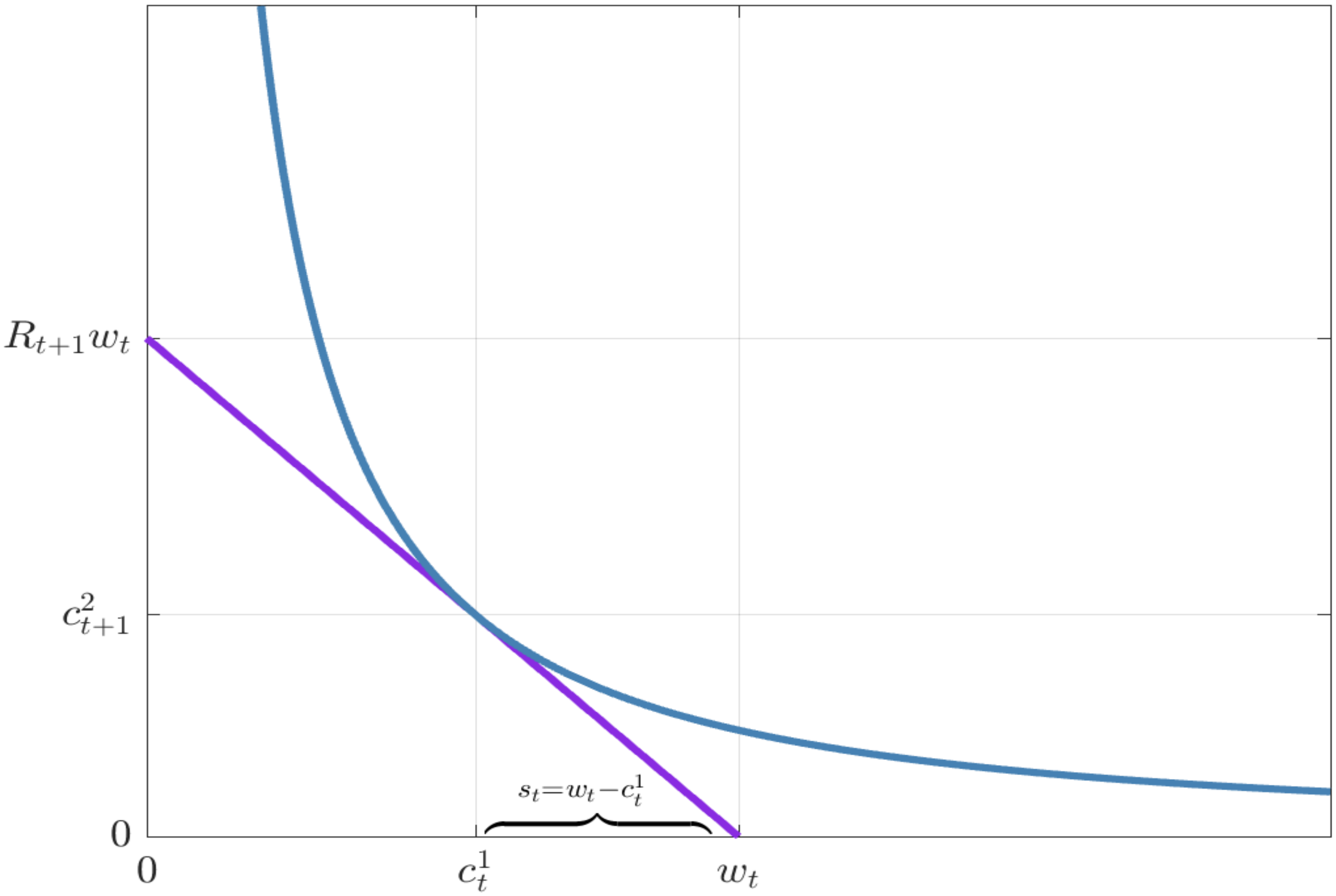
- The first order condition for $s_t$ is a consumption Euler equation

$$u'(c_t^1) = \beta u'(c_{t+1}^2) R_{t+1}$$

- This pins down saving as a function of $w$ and $R$

$$u'(w - s) = \beta u'(Rs)R \qquad \Rightarrow \qquad s(w, R)$$

# Individual Consumption/Savings Problem



# Demographics and Technology

- Let $L_t$ denote mass of *young* individuals at date $t$, growing at rate $n$

$$L_t = (1+n)^t L_0, \qquad n > 0$$

- So in period $t$ the population consists of $L_t$ young and $L_{t-1}$ old.

- Also a mass of 'initial old' at date $t = 0$, endowed with initial assets.

- Aggregate production function

$$Y_t = F(K_t, L_t)$$

  with standard properties, and normalizing $A = 1$.

- To streamline exposition, assume *full depreciation*, $\delta = 1$, so $K_{t+1} = I_t$.

# Factor Prices

- Let $y_t = Y_t/L_t$ denote output per worker, $k_t = K_t/L_t$ etc.
- Intensive form of the production function

$$y_t = f(k_t)$$

- With competitive firms and and competitive factor markets

$$R_t = f'(k_t)$$

$$w_t = f(k_t) - f'(k_t)k_t$$

# Saving and Investment

- Let $S_t$ denote aggregate savings.

- Since there are $L_t$ workers each with savings $s_t$ this is

$$S_t = s_t L_t$$

- Since the economy is closed and there are no government purchases

$$S_t = I_t$$

- Hence

$$K_{t+1} = I_t = S_t = s_t L_t = s(w_t, R_{t+1}) L_t$$

# Goods Market Clearing

- Check that this implies goods market clearing.

- Use of goods by the young at date $t$

$$L_t(c_t^1 + s_t) = L_t w_t = L_t(f(k_t) - f'(k_t)k_t) = Y_t - f'(k_t)K_t$$

- Use of goods by the old at date $t$

$$L_{t-1}c_t^2 = L_{t-1}R_t s_{t-1} = R_t K_t = f'(k_t)K_t$$

- Summing these gives the goods market clearing condition

$$L_t(c_t^1 + s_t) + L_{t-1}c_t^2 = Y_t$$

# Key Equilibrium Condition

- Recall $K_{t+1} = s(w_t, R_{t+1})L_t$.

- Using our expressions for factor prices we get, in per worker terms

$$(1+n)k_{t+1} = s\Big(\underbrace{f(k_t) - f'(k_t)k_t}_{=w_t}, \underbrace{f'(k_{t+1})}_{=R_{t+1}}\Big)$$

- Given $k_t$, look for $k_{t+1}$ that satisfies this equilibrium condition.

- May not be a unique solution to this problem, depends on shape of saving function and production function.

- What can we say about the shape of the saving function $s(w, R)$?

# Outline

1. **Notes on the economics of infinity**

2. **Benchmark two-period OLG growth model**
   - Setup
   - Savings function
   - Equilibrium uniqueness
   - Examples

3. **The possibility of dynamic inefficiency**

4. **Social security and capital accumulation**

# Saving Function $s(w, R)$

- Implicitly determined by

$$u'(w - s) = \beta R u'(Rs)$$

- Savings are *strictly increasing in wage $w$*, can show

$$s_w(w, R) = \frac{1}{1 + \frac{\mathcal{E}(w-s)}{\mathcal{E}(Rs)}\left(\frac{w-s}{s}\right)} \in (0, 1)$$

where $\mathcal{E}(x) > 0$ is the intertemporal elasticity of substitution at $x$.

- But *savings may be increasing or decreasing in return $R$*, can show

$$s_R(w, R) = \frac{\mathcal{E}(Rs) - 1}{1 + \frac{\mathcal{E}(w-s)}{\mathcal{E}(Rs)}\left(\frac{w-s}{s}\right)} \frac{s(w, R)}{R}$$

the sign of which depends on the magnitude of $\mathcal{E}(Rs)$.

# Saving Function $s(w, R)$

- Change in $R$ has both a substitution effect and an income effect on savings
  - increase in $R$ increases the relative price of consumption when young compared to consumption when old and induces substitution away from consumption when young, i.e., increasing saving
  - increase in $R$ increases amount of income when old per unit saving, decreasing the need to save for old age
- Substitution effect dominates if and only if $\mathcal{E}(Rs) > 1$.

# Outline

1. **Notes on the economics of infinity**

2. **Benchmark two-period OLG growth model**
   - Setup
   - Savings function
   - Equilibrium uniqueness
   - Examples

3. **The possibility of dynamic inefficiency**

4. **Social security and capital accumulation**

# Equilibrium Uniqueness

- Write key equilibrium condition

$$(1+n)k_{t+1} = s(w(k_t), f'(k_{t+1}))$$

where the wage

$$w(k_t) \equiv f(k_t) - f'(k_t)k_t$$

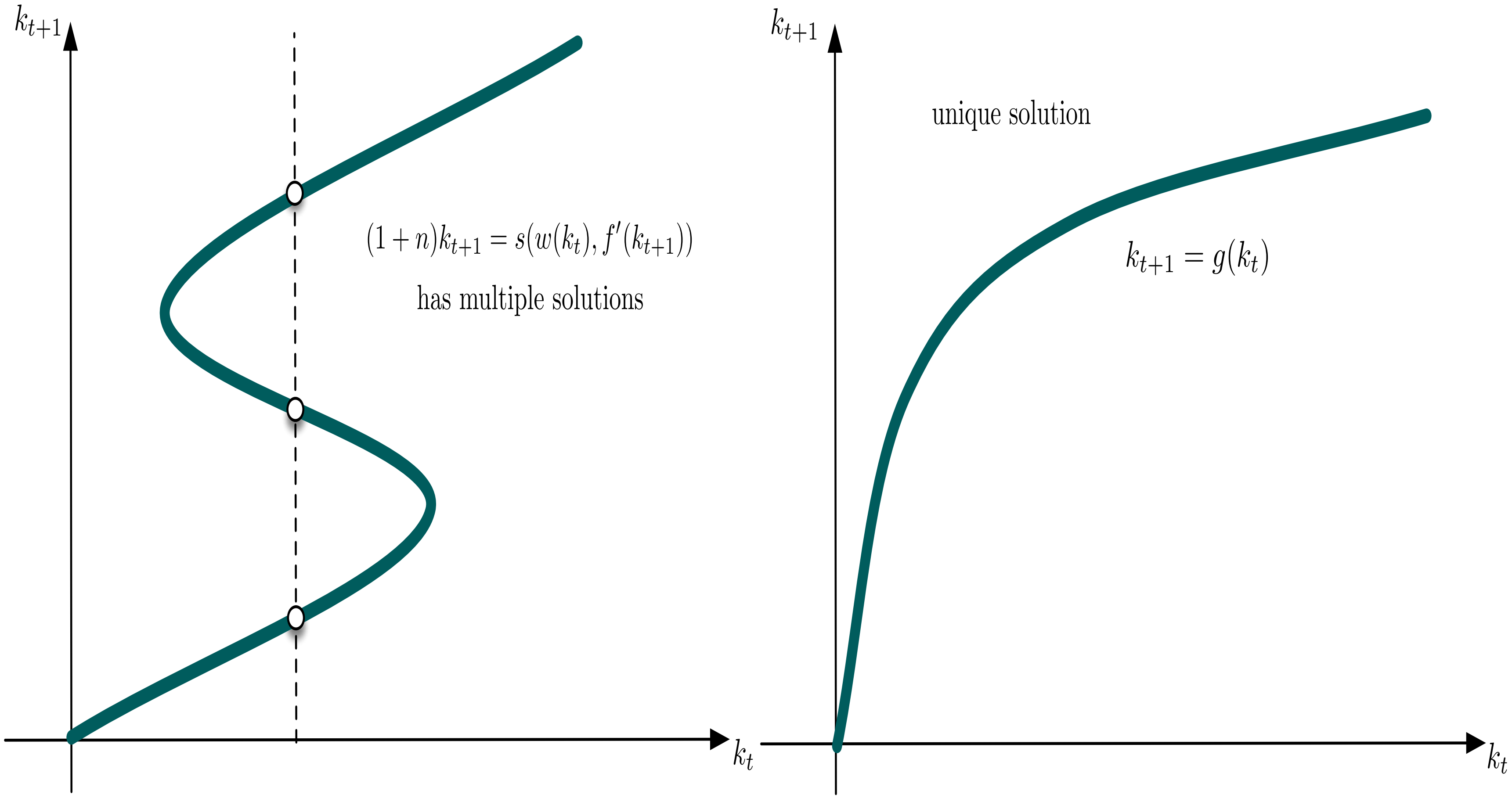
is strictly increasing in $k_t$, so can treat $w_t$ as stand in for $k_t$.

- For given $w_t > 0$, is there a *unique* $k_{t+1}$ that satisfies this equation?

- If there is a unique solution, the dynamics are *determinate* and we can write $k_{t+1} = g(k_t)$ and proceed to study the properties of $g(k)$ to characterize those dynamics.

- If there is a multiplicity of solutions, the dynamics are *indeterminate* and some further equilibrium selection device is required (e.g., '*sunspots*').

# Equilibrium Uniqueness




Case on the left has multiple equilibria, for some $k_t$ there are multiple solutions $k_{t+1}$ to the key equilibrium condition $(1+n)k_{t+1} = s(w(k_t), f'(k_{t+1}))$. Case on the right has a unique equilibrium, for any fixed $k_t$ there is a unique $k_{t+1} = g(k_t)$ solving the equilibrium condition.

# Sufficient Condition for Equilibrium Uniqueness

- Fix $w > 0$ and consider any $k$ such that

$$(1+n)k = s(w, f'(k))$$

- A *sufficient condition* for $k$ to be unique is that

$$s_R(w, f'(k)) > \frac{1+n}{f''(k)}$$

- For example, if $s_R(w, R) > 0$ this condition is satisfied.

- Intuitively, this condition requires that the *income effects* from a change in $R$ are '*not too strong*'.

- But difficult to check in practice, because $w$ and $k$ are endogenous.

- To make further progress, let's consider some specific functional forms.

# Outline

1. **Notes on the economics of infinity**

2. **Benchmark two-period OLG growth model**
   - Setup
   - Savings function
   - Equilibrium uniqueness
   - Examples

3. **The possibility of dynamic inefficiency**

4. **Social security and capital accumulation**

# Specific Functional Forms

- Suppose the period utility function is

$$u(c) = \frac{c^{1-\theta} - 1}{1-\theta}, \qquad \theta > 0$$

with constant intertemporal elasticity of substitution $1/\theta$.

- Suppose the production function is

$$F(K,L) = \left(\alpha K^{\frac{\sigma-1}{\sigma}} + (1-\alpha)L^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}, \qquad 0 < \alpha < 1, \qquad \sigma > 0$$

with constant elasticity of substitution $\sigma$. For future reference

$$f(k) = \left(\alpha k^{\frac{\sigma-1}{\sigma}} + (1-\alpha)\right)^{\frac{\sigma}{\sigma-1}}$$

with

$$f'(k) = \alpha\left(\frac{f(k)}{k}\right)^{\frac{1}{\sigma}}$$

# Individual Consumption/Savings Problem

- With this utility function, the consumption Euler equation is

$$(w-s)^{-\theta} = \beta R (Rs)^{-\theta}$$

which solves for

$$s(w,R) = \frac{\beta^{\frac{1}{\theta}} R^{\frac{1-\theta}{\theta}}}{1+\beta^{\frac{1}{\theta}} R^{\frac{1-\theta}{\theta}}}\, w$$

- Notice this has the anticipated properties

$$s_w(w,R) \in (0,1), \qquad \text{and} \qquad s_R(w,R) > 0 \qquad \Leftrightarrow \qquad \theta < 1$$

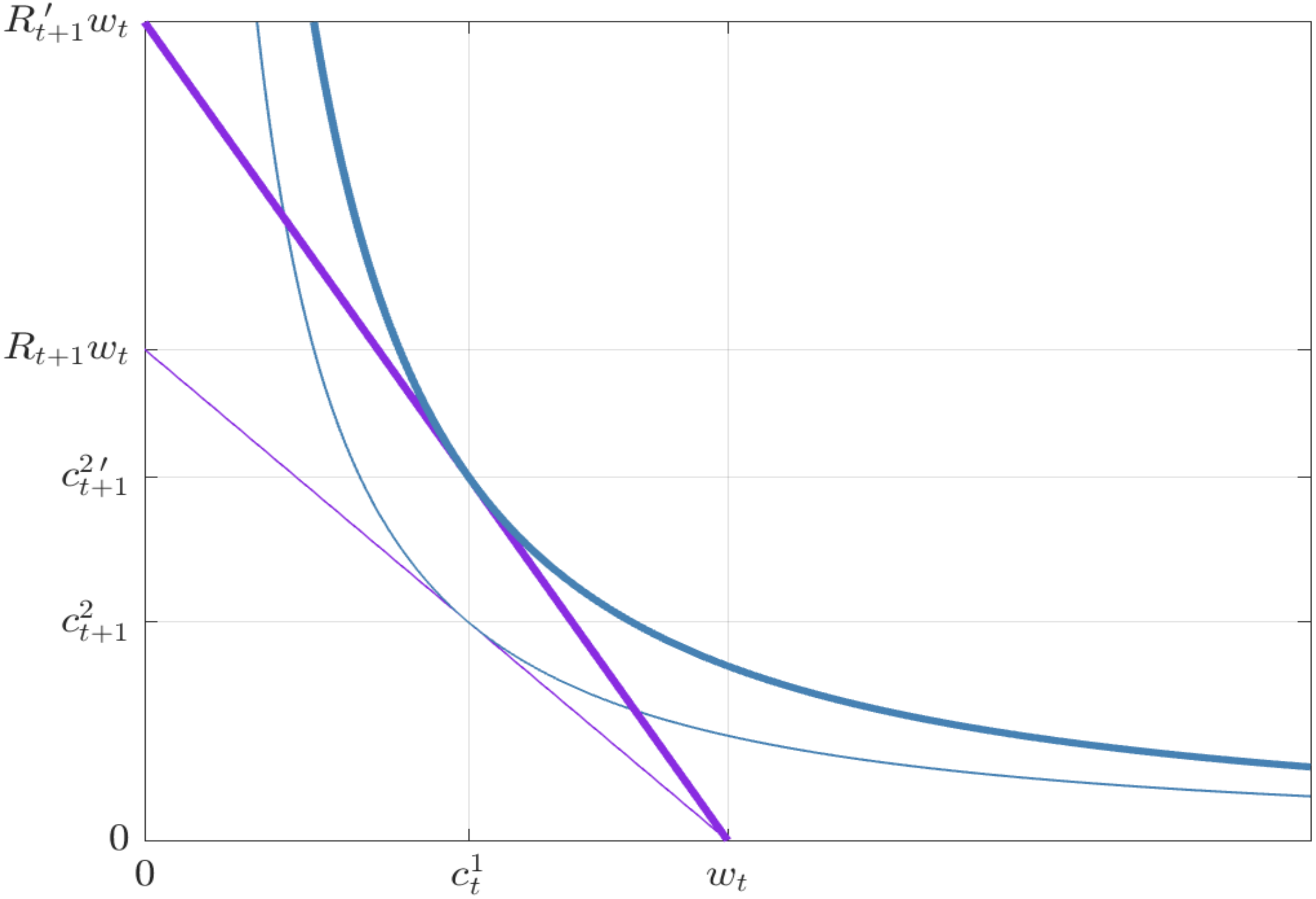
- Can then back out

$$c^1(w,R) = w - s(w,R) = \frac{1}{1+\beta^{\frac{1}{\theta}} R^{\frac{1-\theta}{\theta}}}\, w$$

and

$$c^2(w,R) = Rs(w,R) = \frac{\beta^{\frac{1}{\theta}} R^{\frac{1-\theta}{\theta}}}{1+\beta^{\frac{1}{\theta}} R^{\frac{1-\theta}{\theta}}}\, Rw$$

# Individual Consumption/Savings Problem [$\theta = 1$]



# Back to Equilibrium Uniqueness

- Fix $w > 0$ and consider any $k$ such that

$$(1+n)k = s(w, f'(k))$$

- With these constant elasticity functional forms, this can be written

$$k + k\beta^{-\frac{1}{\theta}} f'(k)^{\frac{\theta-1}{\theta}} = \frac{w}{1+n}$$

  where

$$f'(k) = \alpha\Big(\frac{f(k)}{k}\Big)^{\frac{1}{\sigma}}$$

- Can then show that a *sufficient condition for uniqueness* is

$$\sigma + \frac{1}{\theta} \geq 1$$

- Remarks. The *sum of elasticities* needs to be sufficiently large.

  – if factors are relatively substitutable, $\sigma \geq 1$, satisfied for any $\theta > 0$
  
  – if factors are relatively complementary, $\sigma < 1$, need sufficiently high intertemporal elasticity $1/\theta$

# Equilibrium Dynamics: Overview

- Assumption. Suppose the sufficient condition for uniqueness is satisfied

$$\sigma + \frac{1}{\theta} \geq 1$$

- Under this assumption, there is a unique $k_{t+1} = g(k_t)$ solving

$$(1+n)k_{t+1} = s(w(k_t), f'(k_{t+1}))$$

- Moreover dynamics are *monotone*, the function $g(k)$ is increasing in $k$.

- The function may have *multiple steady states* $k^*$.

- But the dynamics are *bounded*, cannot generate unbounded growth even with '$Ak$' production function.

- Remark. Boundedness results from two key properties

$$(1+n)k_{t+1} = s(w(k_t), f'(k_{t+1})) \leq w(k_t), \qquad \text{and} \qquad \lim_{k\to\infty} \frac{w(k)}{k} = 0$$

# Example: Log / Cobb-Douglas

- The combination of log utility $u(c) = \log c$ (i.e., $\theta = 1$) and Cobb-Douglas production $f(k) = Ak^\alpha$ (i.e., $\sigma = 1$) is particularly straightforward.

- Income and substitution effects of changes in $R$ on savings cancel leaving

$$s(w, R) = \frac{\beta}{1+\beta}\, w, \qquad \text{independent of } R$$

- The wage rate is

$$w(k) = f(k) - f'(k)k = (1-\alpha)Ak^\alpha = (1-\alpha)f(k)$$

- So the equilibrium condition simplifies to

$$(1+n)k_{t+1} = \frac{\beta}{1+\beta}(1-\alpha)Ak_t^\alpha$$

# Example: Log / Cobb-Douglas

- Writing this

$$k_{t+1} = g(k_t) \equiv \frac{(1-\alpha)A\beta}{(1+n)(1+\beta)}\, k_t^{\alpha}$$

  we see that there is a unique non-trivial steady state $k^* > 0$ satisfying $k^* = g(k^*)$ which evaluates to

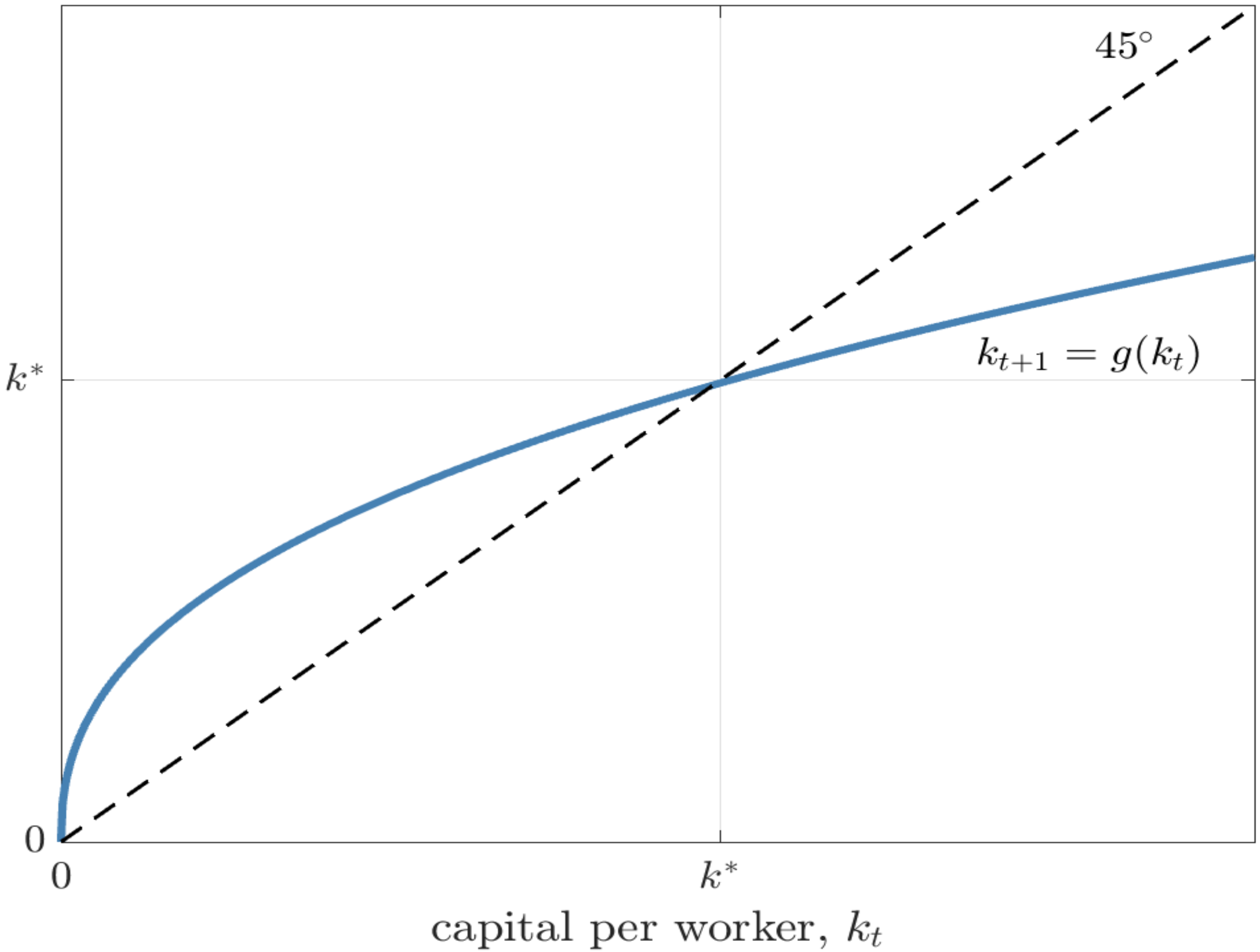
$$k^* = \left(\frac{(1-\alpha)A\beta}{(1+n)(1+\beta)}\right)^{\frac{1}{1-\alpha}}$$

- Qualitatively, the dynamics here are similar to the Solow model.

- But note if $\alpha \to 1$ so that $f(k) = Ak$ we would have $k^* \to 0$ for any $A > 0$.

- Constant saving rate out of *wage income* is not the same as a constant saving rate out of *total income*.

# Example: Log / Cobb-Douglas



# Example: Log / CES

- Suppose log utility $u(c) = \log c$ (i.e., $\theta = 1$) with CES production function

$$f(k) = \left(\alpha k^{\frac{\sigma-1}{\sigma}} + (1-\alpha)\right)^{\frac{\sigma}{\sigma-1}}, \qquad \sigma > 0$$

  for which the wage is

$$w(k) = (1-\alpha)\left(\alpha k^{\frac{\sigma-1}{\sigma}} + (1-\alpha)\right)^{\frac{1}{\sigma-1}} = (1-\alpha) f(k)^{\frac{1}{\sigma}}$$

- So in this case

$$k_{t+1} = g(k_t) \equiv \frac{(1-\alpha)\beta}{(1+n)(1+\beta)} f(k_t)^{\frac{1}{\sigma}}$$

- Production function $f(k)$ is concave, but what about $g(k)$?

- Differentiating twice and collecting terms we get

$$\frac{g''(k)}{g(k)} = \frac{1}{\sigma}\left\{\frac{f''(k)}{f(k)} + \left(\frac{1}{\sigma} - 1\right)\left(\frac{f'(k)}{f(k)}\right)^2\right\}$$

# Example: Log / CES

- From this we have two cases to consider.

  **(i)** *factors are relatively substitutable*, $\sigma \geq 1$, implies $g''(k) < 0$ for all $k$ and there is a unique non-trivial steady state $k^* > 0$.

  **(ii)** *factors are relatively complementary*, $\sigma < 1$, implies $g''(k) > 0$ for $k < k_{\text{CRIT}}$ and $g''(k) < 0$ for $k > k_{\text{CRIT}}$ where the critical value $k_{\text{CRIT}}$ solves $g''(k) = 0$ and works out to be

$$k_{\text{CRIT}} = \left(\frac{(1-\alpha)}{\alpha(1-\sigma)}\right)^{\frac{\sigma}{\sigma-1}}, \qquad \sigma < 1$$

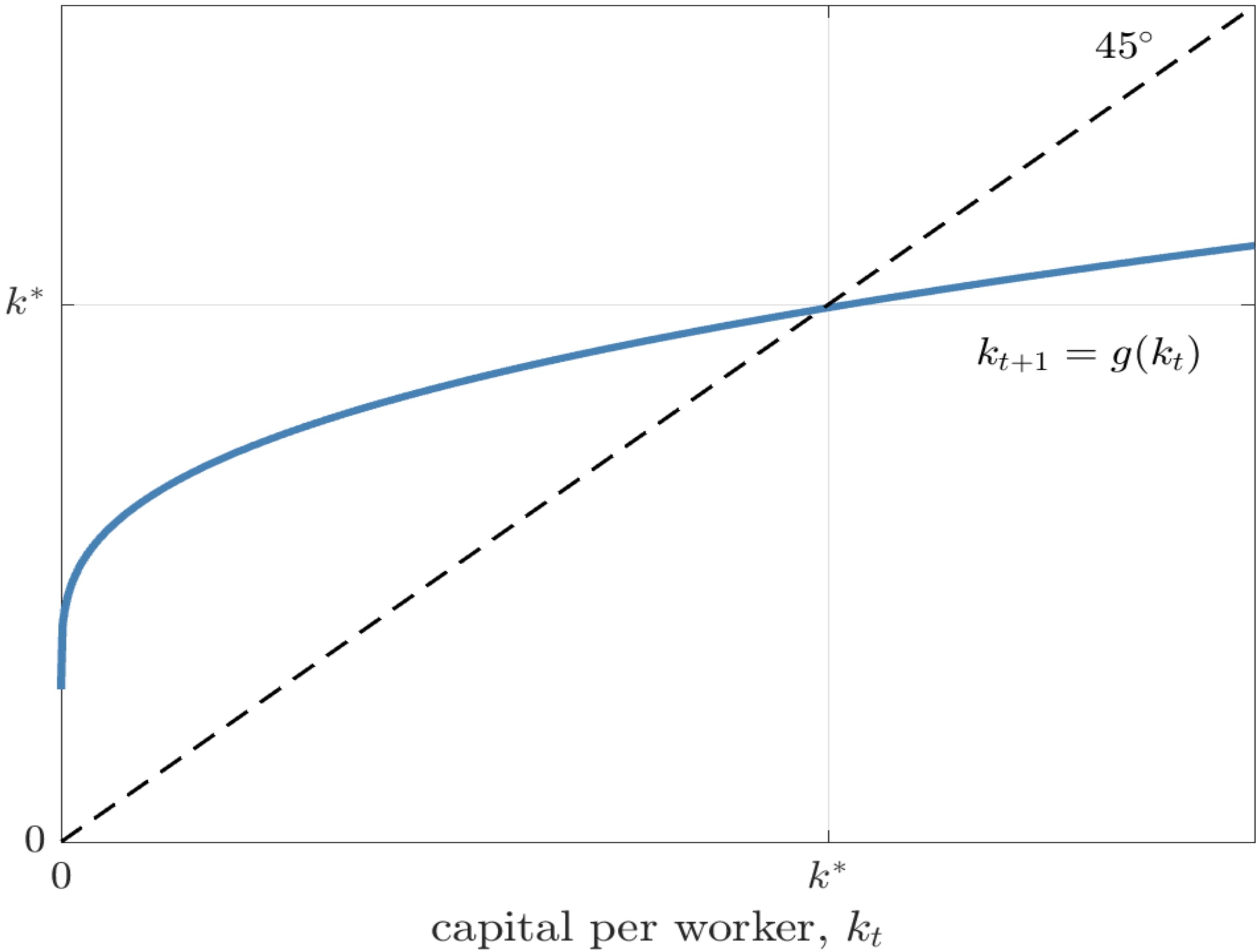
  in this case, there are either

  **(a)** two non-trivial steady states, of which *only the larger is stable*, or

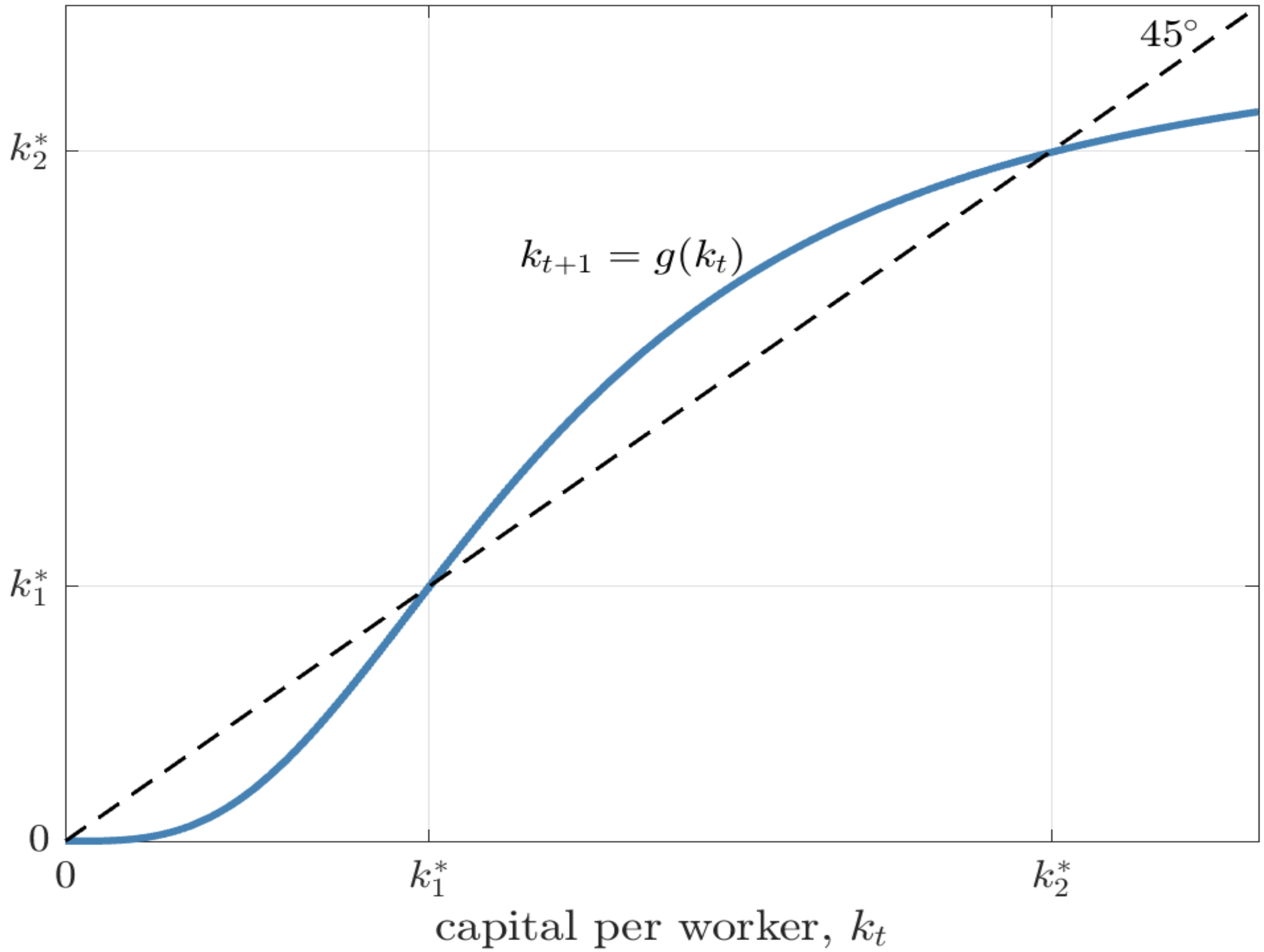
  **(b)** zero non-trivial steady states, if $g(k) < k$ for all $k$

- Remark. In case (ii) there is an endogenous *poverty trap* in the sense that if $k_t < k_{\text{CRIT}}$ then $k_t \to 0^+$.

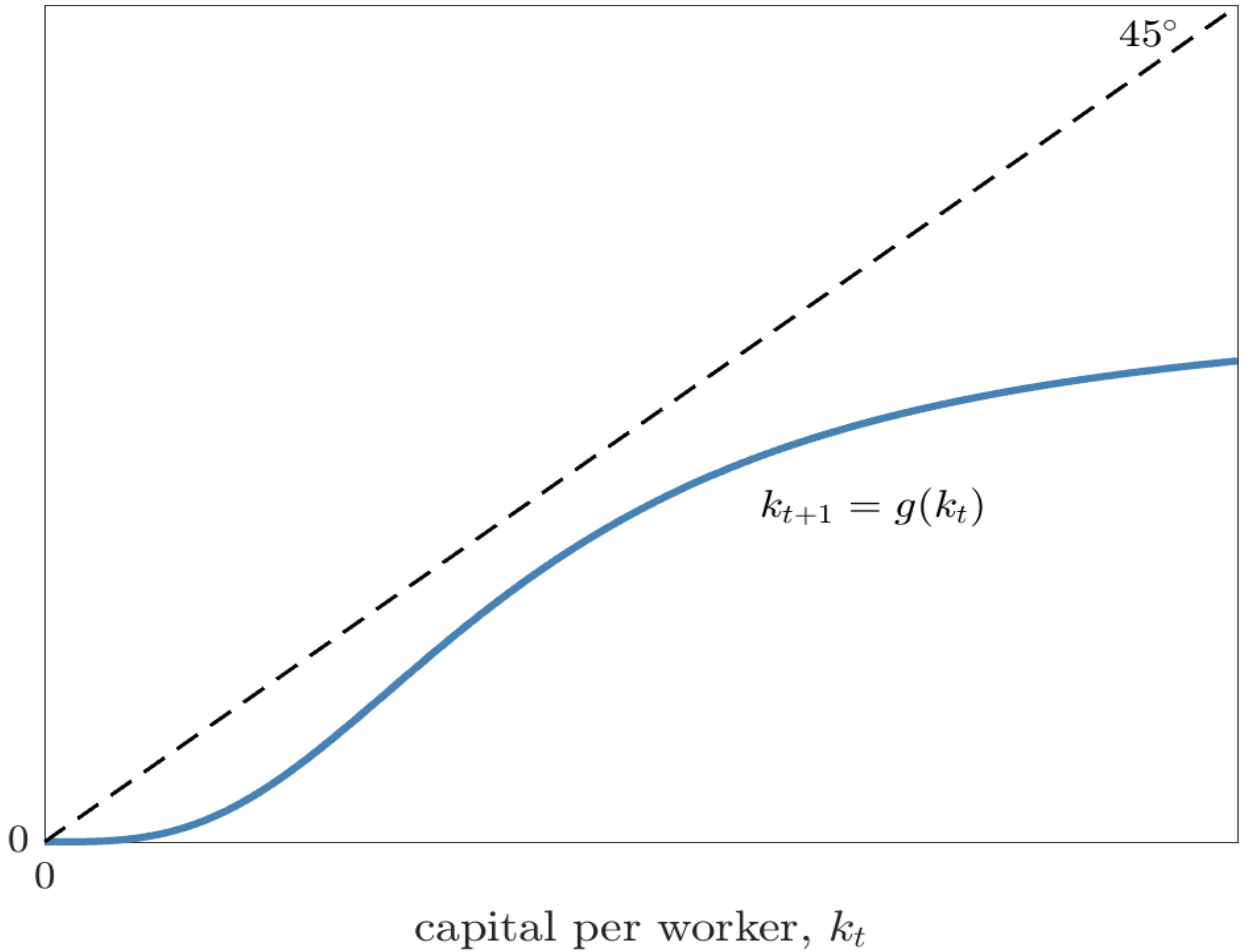
# Example: Log / CES [case (i), $\sigma > 1$]



# Example: Log / CES [case (ii.a), $\sigma < 1$]



# Example: Log / CES [case (ii.b), $\sigma < 1$]



# Outline

1. **Notes on the economics of infinity**

2. **Benchmark two-period OLG growth model**
   - Setup
   - Savings function
   - Equilibrium uniqueness
   - Examples

3. **The possibility of dynamic inefficiency**

4. **Social security and capital accumulation**

# OLG Planning Problem

- Consider planner that seeks to maximize

$$W = \sum_t \omega_t U_t = \sum_t \omega_t \big[u(c_t^1) + \beta u(c_{t+1}^2)\big]$$

subject to sequence of aggregate resource constraints

$$L_t c_t^1 + L_{t-1} c_t^2 + K_{t+1} \leq F(K_t, L_t)$$

- In per worker terms, the resource constraint is

$$c_t^1 + \frac{1}{1+n} c_t^2 + (1+n) k_{t+1} \leq f(k_t)$$

- Remark. Planning weights $\boldsymbol{\omega}$ need only ensure objective is well-defined, do not need $\omega_t = \beta^t$ or indeed need strictly geometric discounting at all.

# OLG Planning Problem

- Lagrangian with multiplier $\lambda_t \geq 0$ for each resource constraint

$$\mathcal{L} = \sum_{t=0}^{\infty} \omega_t \left[ u(c_t^1) + \beta u(c_{t+1}^2) \right] + \sum_{t=0}^{\infty} \lambda_t \left[ f(k_t) - c_t^1 - \frac{1}{1+n} c_t^2 - (1+n) k_{t+1} \right]$$

- Key first order conditions, hold at each date

$$c_t^1 : \qquad \omega_t u'(c_t^1) - \lambda_t = 0$$

$$c_t^2 : \qquad \omega_{t-1} \beta u'(c_t^2) - \lambda_t \frac{1}{1+n} = 0$$

$$k_{t+1} : \qquad -\lambda_t (1+n) + \lambda_{t+1} f'(k_{t+1}) = 0$$

$$\lambda_t : \qquad f(k_t) - c_t^1 - \frac{1}{1+n} c_t^2 - (1+n) k_{t+1} = 0$$

# Intertemporal Consumption Allocation

- Consider intertemporal consumption for individual born at date $t$.

- First order condition for consumption $c_t^1$ when they are young

$$\lambda_t = \omega_t u'(c_t^1)$$

- First order condition for consumption $c_{t+1}^2$ when they are old

$$\lambda_{t+1} = \omega_t \beta u'(c_{t+1}^2)(1+n)$$

- Then using the first order condition for capital accumulation

$$\lambda_t(1+n) = \lambda_{t+1} f'(k_{t+1})$$

we see that the planning weights $\omega_t$ and $1+n$ factors cancel, giving usual

$$u'(c_t^1) = \beta u'(c_{t+1}^2) f'(k_{t+1})$$

just as they would choose for themselves.

# Intratemporal Allocation Between Young & Old

- Now consider intratemporal allocation between young and old at date $t$.

- Comparing the first order conditions for $c_t^1$ and $c_t^2$ we get

$$\omega_t u'(c_t^1) = \omega_{t-1} \beta u'(c_t^2)(1+n)$$

- This condition is static up to the *exogenous* planning weights $\omega_t/\omega_{t-1}$.

- Example. Suppose the planning weights are $\omega_t = \hat{\beta}^t$ for some discount factor $\hat{\beta}$ not necessarily equal to $\beta$. Then the planner would set

$$\frac{u'(c_t^2)}{u'(c_t^1)} = (1+n)\frac{\hat{\beta}}{\beta}$$

so that the planner trades off consumption for the old vs. consumption for the young at an effective relative price of $(1+n)\hat{\beta}/\beta$.

# Discussion

- Conditional on $R_{t+1} = f'(k_{t+1})$, the planner allocates individual lifetime consumption exactly as the individuals would choose for themselves.

- This is because there is no 'market failure' for the planner to correct, *conditional on $R_{t+1} = f'(k_{t+1})$.*

- Note this is independent of the planning weights $\omega_t$. Instead, where the planning weights matter is in allocating resources between young and old within a given period.

- So key question becomes, how does planner's $k_{t+1}$ and hence planner's $R_{t+1} = f'(k_{t+1})$ compare to decentralized market outcome?

- To figure this out, we need to see how planner's $k_{t+1}$ is determined.

# Dynamical System

- System of three nonlinear difference equations in $c_t^1, c_t^2$ and $k_{t+1}$

  $$u'(c_t^1) = \beta u'(c_{t+1}^2) f'(k_{t+1})$$

  and

  $$c_t^1 + \frac{1}{1+n} c_t^2 + k_{t+1} = f(k_t)$$

  and

  $$\omega_t u'(c_t^1) = \omega_{t-1} \beta u'(c_t^2)(1+n)$$

  taking as given the planning weights $\omega_t / \omega_{t-1}$ which drive any time-dependence in the allocation between $c_t^1$ and $c_t^2$.

- Assumption. Suppose planning weights are asymptotically geometric

  $$\lim_{t\to\infty} \frac{\omega_t}{\omega_{t-1}} = \hat{\beta} \in (0,1)$$

  for some $\hat{\beta}$ not necessarily equal to $\beta$.

# Steady State

- Steady state $c^{1*}$, $c^{2*}$ and $k^*$.

- Would ordinarily start with Euler equation

$$u'(c_t^1) = \beta u'(c_{t+1}^2) f'(k_{t+1})$$

- But in general $c^{1*} \neq c^{2*}$ so *cannot* use individual Euler equation to conclude planner's $k^*$ solves $1 = \beta f'(k^*)$.

- Instead use intratemporal allocation between young and old

$$\hat{\beta} u'(c_t^1) = \beta u'(c_t^2)(1+n)$$

to write Euler equation between the young at $t$ and the young at $t+1$ as

$$u'(c_t^1) = \frac{\hat{\beta}}{1+n} u'(c_{t+1}^1) f'(k_{t+1})$$

# Steady State

- So planner's steady state capital stock solves

$$\hat{\beta} f'(k^*) = 1 + n$$

- This is in general different to the decentralized market outcome.

- Given $k^*$, aggregate consumption per worker $c^*$ is then given by

$$c^* \equiv c^{1*} + \frac{1}{1+n} c^{2*} = f(k^*) - (1+n)k^*$$

- We then split $c^*$ into $c^{1*}$ and $c^{2*}$ using the intratemporal allocation between young and old, namely

$$\hat{\beta} u'(c^{1*}) = \beta u'(c^{2*})(1+n)$$

# Golden Rule

- As in the basic Solow model, aggregate consumption per worker is 'hump-shaped' in $k^*$.

- In particular

$$\frac{dc^*}{dk^*} = f'(k^*) - (1+n) < 0 \qquad \Leftrightarrow \qquad k^* < k^*_{\text{GR}}$$

where the golden rule level is given by

$$f'(k^*_{\text{GR}}) = 1 + n$$

(Approximately the same as the planner's steady state if $\hat{\beta} \approx 1$)

- Let $k^*_{\text{CE}}$ denote the competitive equilibrium steady state capital stock.

- If $k^*_{\text{CE}} > k^*_{\text{GR}}$ then we can increase consumption for young *and* old, thereby making both better off, by *reducing saving*, i.e., reducing capital.

# Dynamic Inefficiency

- We say that the economy is *dynamically inefficient* if $k^*_{\text{CE}} > k^*_{\text{GR}}$.

- Since $k^*_{\text{CE}}$ satisfies $f'(k^*_{\text{CE}}) = R^*$ and $k^*_{\text{GR}}$ satisfies $f'(k^*_{\text{GR}}) = 1 + n$, equivalently an economy is dynamically inefficient if

$$r^* < n$$

  where $r^* = R^* - 1$ is the net real return to capital.

- REMARK. This configuration was impossible in the counterpart neoclassical growth model, which has $r^* > \rho + n$ where $\rho = 1/\beta - 1$.

# Intuition

- Young at time $t$ face prices $w_t, R_t$ reflecting the capital stock $k_t$.

- Capital stock $k_t$ the result of previous generations savings decisions.

- In other words, previous generations' savings decisions impose a *pecuniary externality* on the current (and future) young.

- Ordinarily, pecuniary externalities do not cause an equilibrium to be Pareto inefficient, i.e., are not a source of market failure.

- But here there is a perpetual arrival of new young, $n > 0$, and the planner *may* be able to rearrange consumption over time to take advantage of these pecuniary externalities.

# Alternative Intuition

- Dynamic inefficiency results from overaccumulation of capital.
- Saving results from young providing for their old age. Young will have a strong incentive to save if they have a declining lifetime labor income profile, e.g., the $(w_t, 0)$ profile here.
- But the more they young save, the lower is the return on capital.
- This creates an adverse income effect, which, if strong enough, only encourages more saving.
- If only there was another vehicle for saving which did not depress the return on physical capital!

# Outline

1. **Notes on the economics of infinity**

2. **Benchmark two-period OLG growth model**
   - Setup
   - Savings function
   - Equilibrium uniqueness
   - Examples

3. **The possibility of dynamic inefficiency**

4. **Social security and capital accumulation**

# Social Security and Capital Accumulation

- Social security provides a possible solution to overaccumulation.

- We will contrast two extremes

  **(i)** *fully-funded system*, young make contributions to social security system, paid back to them in old age

  **(ii)** *unfunded system*, transfers from young to old, making use of perpetual arrival of new young

  discourages saving, but that may be Pareto-improving

# Fully-Funded System

- Government takes $d_t$ from young workers, invested in physical capital and returned, with interest, when old.

- Taking $d_t$ and factor prices as given, individual born at date $t$ chooses consumption and savings $s_t$ to maximize

$$U_t = u(c_t^1) + \beta u(c_{t+1}^2), \qquad 0 < \beta < 1$$

subject to

$$c_t^1 + s_t + d_t \leq w_t$$

and

$$c_{t+1}^2 \leq R_{t+1}(s_t + d_t)$$

- Capital per worker now evolves according to

$$(1+n)k_{t+1} = s_t + d_t$$

# Fully-Funded System

- Previously only way to have $c^2_{t+1} > 0$ was to have $s_t > 0$, so workers had to save themselves.

- Now young workers effectively choose $\hat{s}_t \equiv s_t + d_t$ and our previous analysis goes through [this may require $s_t < 0$ if $d_t$ is large relative to the savings young would choose if $d_t = 0$].

- In other words, taking $d_t$ as given young households choose savings $s_t$ that perfectly offset $d_t$ so that they end up with the consumption/savings profiles they would have had if $d_t = 0$.

- In this sense, a fully-funded system cannot address the overaccumulation problem, if it exists.

# Unfunded System

- Government takes $d_t$ from each of $L_t$ young, transfers to *current old* giving them $d_t L_t / L_{t-1} = (1+n)d_t$ each.

- Taking $d_t$ and factor prices as given, individual born at date $t$ chooses consumption and savings $s_t$ to maximize

$$U_t = u(c_t^1) + \beta u(c_{t+1}^2), \qquad 0 < \beta < 1$$

subject to

$$c_t^1 + s_t + d_t \leq w_t$$

and

$$c_{t+1}^2 \leq R_{t+1} s_t + (1+n) d_{t+1}$$

- Capital per worker now evolves according to

$$(1+n)k_{t+1} = s_t$$

because here $d_t$ is a within-period transfer, not invested in capital.

# Unfunded System

- Rate of return on social security payments is $n$, not $r_{t+1}$.

- Income effect of payments $(1+n)d_{t+1}$ discourages saving at the margin.

- Would be unfortunate if economy is dynamically efficient, because would decrease capital formation and decrease consumption of young and old.

- But *may* be Pareto-improving if economy is dynamically inefficient, lessens the overaccumulation problem.

- Initial old are *windfall beneficiaries*, receiving transfers from initial young never having made contributions themselves.

# Summary

- OLG provides a tractable alternative to neoclassical growth model.

- In special cases, looks just like the Solow growth model. But much richer dynamics are possible, especially if income effects are strong or factors are sufficiently complementary.

- Perpetual arrival of new young creates a ‘double infinity’ of agents and commodities.

- Competitive equilibrium may be inefficient, even absent traditional sources of market failure.

- Economy may be dynamically inefficient, accumulating too much capital.

- But probably should not over-emphasize dynamic inefficiency. For most countries the problem seems to be too little capital not too much.

# Next Class

- Endogenous growth.

- Externalities in capital accumulation.

- Variations on the $Ak$ theme.

# Homework

- Consider the two-period OLG model and suppose the utility and production functions are

$$u(c) = \frac{c^{1-\theta} - 1}{1-\theta}, \qquad \text{and} \qquad f(k) = \left(\alpha k^{\frac{\sigma-1}{\sigma}} + (1-\alpha)\right)^{\frac{\sigma}{\sigma-1}}$$

- Suppose the sufficient condition for equilibrium uniqueness is satisfied

$$\sigma + \frac{1}{\theta} \geq 1$$

- Let $k_{t+1} = g(k_t)$ solve the key equilibrium condition.

- Check. Show that the equilibrium dynamics are *montone*, $g'(k) > 0$.

- Check. Show that the equilibrium dynamics are *bounded*, for any $k_0 > 0$ we have $k_t \leq \max[k_0, \bar{k}]$ for some $\bar{k} < \infty$ to be determined.