Economic Growth

Lecture 7: Neoclassical growth model, part three

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Outline

Representative Agent

- Till now we have simply assumed all households identical.
- How should we understand the *representative agent* construct when households not identical? Two main interpretations:

  (i) *weaker, positive interpretation*

  outcomes are such that it is 'as if' there is a single decision maker (i.e., a single objective function and single constraint set)

  (ii) *stronger, normative interpretation*

  households are sufficiently similar that the objective function of the single decision maker can be used for welfare analysis

  does the representative household have the same discount factor? attitudes to intertemporal substitution? labor supply? risk aversion?

Setup

- Consider an *exchange economy.*

- Individuals $i = 1, 2, \ldots, I$ have endowments $\boldsymbol{y}_i$ and face prices $\boldsymbol{p}$.

- Let $\boldsymbol{c}_i(\boldsymbol{p}, \boldsymbol{y}_i)$ denote their individual demands

$$\boldsymbol{c}_i(\boldsymbol{p}, \boldsymbol{y}_i) \equiv \operatorname*{argmax}_{\boldsymbol{c} \geq 0} u_i(\boldsymbol{c}) \quad \text{subject to} \quad \boldsymbol{p} \cdot \boldsymbol{c} \leq \boldsymbol{p} \cdot \boldsymbol{y}_i$$

- Let $\boldsymbol{x}_i(\boldsymbol{p}, \boldsymbol{y}_i) \equiv \boldsymbol{c}_i(\boldsymbol{p}, \boldsymbol{y}_i) - \boldsymbol{y}_i$ denote their individual excess demands.

- Let $\bar{\boldsymbol{x}}(\boldsymbol{p})$ denote aggregate (or average) excess demand

$$\bar{\boldsymbol{x}}(\boldsymbol{p}) = \frac{1}{I} \sum_{i=1}^{I} \boldsymbol{x}_i(\boldsymbol{p}, \boldsymbol{y}_i)$$

- What can we say about $\bar{\boldsymbol{x}}(\boldsymbol{p})$? How might it represent the underlying preferences $\boldsymbol{x}_i(\boldsymbol{p})$? How does $\bar{\boldsymbol{x}}(\boldsymbol{p})$ depend on the distribution of $\boldsymbol{y}_i$?

Sonnenschein-Mantel-Debreu

- A standard result in general equilibrium theory suggests we should not expect to be able to say much.

- Let $\boldsymbol{f}(\boldsymbol{p})$ be a function that is (i) continuous, (ii) homogeneous of degree zero in prices $\boldsymbol{p}$, and (iii) satisfies Walras' law, $\boldsymbol{p} \cdot \boldsymbol{f}(\boldsymbol{p}) = 0$.

- Proposition (Sonnenschein 1972, Mantel 1974, Debreu 1974). There exists an exchange economy with aggregate excess demand $\bar{\boldsymbol{x}}(\boldsymbol{p}) = \boldsymbol{f}(\boldsymbol{p})$.

- Individual optimization, by itself, places almost no restrictions on the shape of the aggregate excess demand.

Sonnenschein-Mantel-Debreu

- Put differently, 'aggregate preferences' encoded in the aggregate excess demand $\bar{\boldsymbol{x}}(\boldsymbol{p})$ may be fundamentally different — *not representative of* — the actual preferences of the true individual decision makers.

- For example, individual optimality requires decisions satisfy the weak axiom of revealed preference, but the aggregate $\bar{\boldsymbol{x}}(\boldsymbol{p})$ need not.

- We need (a lot) more structure on the underlying primitives before we can hope to have useful aggregation.

- Let's now look at a case that, by contrast, aggregates perfectly.

Outline

1. **Aggregation and the representative agent**
   Gorman aggregation

2. **Welfare theorems**
   First welfare theorem
   Second welfare theorem

3. **Dynamic general equilibrium theory with complete markets**
   Pareto problem
   Arrow-Debreu problem
   Examples
   More heterogeneity

Gorman Aggregation

- In slight abuse of notation, now let y_i denote individual *income*.
- Let $v_i(\boldsymbol{p}, y_i)$ denote the indirect utility of individual i

$$v_i(\boldsymbol{p}, y_i) \equiv \max_{\boldsymbol{c} \geq 0} u_i(\boldsymbol{c}) \quad \text{subject to} \quad \boldsymbol{p} \cdot \boldsymbol{c} \leq y_i$$

- Recall that $v_i(\boldsymbol{p}, y_i)$ is homogeneous of degree zero in $(\boldsymbol{p}, y_i)$.
- Proposition (Gorman 1961). If the preferences of individual i can be represented by an indirect utility function of the form

$$v_i(\boldsymbol{p}, y_i) = a_i(\boldsymbol{p}) + b(\boldsymbol{p})y_i$$

  where $a_i(\boldsymbol{p})$ is homogenous of degree zero and $1/b(\boldsymbol{p})$ is homogeneous of degree one, then the representative agent has preferences

$$\bar{v}(\boldsymbol{p}, \bar{y}) = \bar{a}(\boldsymbol{p}) + b(\boldsymbol{p})\bar{y}$$

  where $\bar{y}$ denotes the average of y_i across i etc.

Gorman Aggregation

- Proof (Sketch). Let $e_i(\boldsymbol{p}, u)$ denote the associate expenditure function, implicitly defined by $v_i(\boldsymbol{p}, e) = u$. With the Gorman form

$$e_i(\boldsymbol{p}, u) = \frac{u - a_i(\boldsymbol{p})}{b(\boldsymbol{p})}$$

By the envelope theorem, the demand for any good j by individual i is

$$c_{ij}(\boldsymbol{p}, y_i) = \frac{\partial}{\partial p_j} e_i(\boldsymbol{p}, u) = -\frac{1}{b(\boldsymbol{p})} \frac{\partial a_i(\boldsymbol{p})}{\partial p_j} - \frac{1}{b(\boldsymbol{p})} \frac{\partial b(\boldsymbol{p})}{\partial p_j} y_i$$

That is, the *Engel curves* are linear. Averaging over i we get

$$\bar{c}_j(\boldsymbol{p}, \bar{y}) = -\frac{1}{b(\boldsymbol{p})} \frac{\partial \bar{a}(\boldsymbol{p})}{\partial p_j} - \frac{1}{b(\boldsymbol{p})} \frac{\partial b(\boldsymbol{p})}{\partial p_j} \bar{y}$$

Which is the same as the demand curve of an individual with preferences represented by $\bar{v}(\boldsymbol{p}, \bar{y}) = \bar{a}(\boldsymbol{p}) + b(\boldsymbol{p})\bar{y}$.

Gorman Aggregation: Discussion

- Crucial that coefficient on income $b(\boldsymbol{p})$ is *the same* for all i.

- Given this representation, only the average income $\bar{y}$ not its whole distribution y_i matters.

- Given this representation, welfare conclusions drawn from the representative household do on average reflect the underlying welfare of households i, i.e., we have the stronger, normative interpretation.

- But even this may not be enough.

- Example. For *homothetic* preferences, indirect utility can be written

$$v_i(\boldsymbol{p}, y_i) = v_i(\boldsymbol{p}, 1)y_i$$

- So homothetic utility satisfies the Gorman form if and only if it is *identically homothetic*, $v_i(\boldsymbol{p}, 1) = v(\boldsymbol{p}, 1)$ for all i.

Aggregation and the Welfare Theorems

- Close connection between aggregation and the welfare theorems.

- Under relatively weak conditions, the *first welfare theorem* says that any competitive equilibrium is Pareto efficient.

- Under stronger conditions, the *second welfare theorem* says that any Pareto efficient allocation can be supported as a competitive equilibrium, using an appropriate set of *transfers*.

- Since a Pareto efficient allocation corresponds to the solution of an optimization problem, the second welfare theorem gives us an *indirect* way to construct a ‘representative agent’.

- But preferences of the representative agent constructed this way may not straightforwardly correspond to preferences of underlying decision makers.

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Exchange Economy

- Dated commodities $t = 0, 1, 2, \ldots$
- Individuals $i = 1, 2, \ldots, I$. For now, I finite.
- Individual endowments $\boldsymbol{y}_i$ with typical element $y_{it} > 0$.
- Individual allocation $\boldsymbol{c}_i$ with typical element c_{it}.
- An allocation $\boldsymbol{c} = \{\boldsymbol{c}_i\}$ is *feasible* if $c_{it} \geq 0$ for every i and t and

$$\sum_i c_{it} \leq \sum_i y_{it}, \qquad \text{for all } t$$

- Prices $\boldsymbol{p}$ with typical element $p_t \geq 0$.

Optimization

- Individual preferences represented by *utility functions* $u_i(\boldsymbol{c}_i)$.

- Individual budget set $\mathcal{B}_i(\boldsymbol{p})$ defined by

$$\mathcal{B}_i(\boldsymbol{p}) \equiv \{\boldsymbol{c}_i \geq 0 \mid \boldsymbol{p} \cdot \boldsymbol{c}_i \leq \boldsymbol{p} \cdot \boldsymbol{y}_i\}$$

- Individual allocation $\boldsymbol{c}_i$ is *optimal* for i if

$$\boldsymbol{c}_i \in \underset{\boldsymbol{c}_i' \in \mathcal{B}_i(\boldsymbol{p})}{\operatorname{argmax}} \; u_i(\boldsymbol{c}_i')$$

- Assumption. Preferences are *locally non-satiated*, implying $\boldsymbol{p} \cdot \boldsymbol{c}_i = \boldsymbol{p} \cdot \boldsymbol{y}_i$.

- Remark. But, for now, have not assumed preferences are convex.

Equilibrium

- A *competitive equilibrium* is a feasible allocation $\boldsymbol{c}$ and prices $\boldsymbol{p}$ such that

  (i) taking $\boldsymbol{p}$ as given, $\boldsymbol{c}_i$ is optimal for each i

  (ii) markets clear

$$\sum_i c_{it} = \sum_i y_{it}, \qquad \text{for all } t$$

Efficiency

- Let $\mathcal{F}$ denote the set of feasible allocations.

- A feasible allocation $\boldsymbol{c}$ is *Pareto dominated* if there exists a feasible allocation $\boldsymbol{c}'$ such that $u_i(\boldsymbol{c}_i') > u_i(\boldsymbol{c}_i)$ for all i.

- Let $\mathcal{D}$ denote the set of Pareto dominated allocations.

- An allocation $\boldsymbol{c}$ is *Pareto efficient* if $\boldsymbol{c} \in \mathcal{F} \backslash \mathcal{D}$.

- Remark. That is, the Pareto efficient set is the set of feasible allocations after all the Pareto dominated allocations have been removed.

Outline

First Welfare Theorem

- Let $\mathcal{P}$ denote the set of *Pareto efficient* allocations.
- Let $\mathcal{C}$ denote the set of competitive equilibria. If $(\boldsymbol{c}, \boldsymbol{p}) \in \mathcal{C}$ we say that $\boldsymbol{c}$ is a *competitive equilibrium allocation*.
- First Welfare Theorem. If $(\boldsymbol{c}, \boldsymbol{p}) \in \mathcal{C}$ then $\boldsymbol{c} \in \mathcal{P}$.
- That is, every competitive equilibrium allocation is Pareto efficient.
- Proof (Sketch). Suppose $(\boldsymbol{c}, \boldsymbol{p}) \in \mathcal{C}$ but $\boldsymbol{c} \notin \mathcal{P}$. Then $\boldsymbol{c} \in \mathcal{D}$ so there exists $\boldsymbol{c}' \in \mathcal{F}$ such that $u_i(\boldsymbol{c}_i') > u_i(\boldsymbol{c}_i)$ for all i. But then $\boldsymbol{p} \cdot \boldsymbol{c}_i' > \boldsymbol{p} \cdot \boldsymbol{c}_i$ $= \boldsymbol{p} \cdot \boldsymbol{y}_i$ for all i, otherwise i would choose $\boldsymbol{c}_i'$ over $\boldsymbol{c}_i$. But if so, summing over i, we get
$$\sum_i \boldsymbol{p} \cdot (\boldsymbol{c}_i' - \boldsymbol{y}_i) = \boldsymbol{p} \cdot \sum_i (\boldsymbol{c}_i' - \boldsymbol{y}_i) > 0$$
which contradicts $\boldsymbol{c}' \in \mathcal{F}$.

First Welfare Theorem: Discussion

- If another allocation was to Pareto dominate the competitive equilibrium allocation, it must not be budget-affordable at the equilibrium prices.

- Implicit in this argument is that the relevant sums are well-defined. To see why, recall that the inner product is

$$\boldsymbol{p} \cdot \boldsymbol{y}_i = \sum_{t=0}^{\infty} p_t y_{it}$$

- So writing, for example

$$\sum_i \boldsymbol{p} \cdot \boldsymbol{y}_i = \boldsymbol{p} \cdot \sum_i \boldsymbol{y}_i$$

is asserting that we can interchange the order of summation

$$\sum_i \sum_{t=0}^{\infty} p_t y_{it} = \sum_{t=0}^{\infty} p_t \sum_i y_{it}$$

- In *overlapping generations* models, first welfare theorem *may* not hold because, as we will see, value of endowments can be '$\sum_t p_t \sum_i y_{it} = +\infty$'.

Outline

Second Welfare Theorem

- Let $(\boldsymbol{c}, \boldsymbol{p}) \in \mathcal{C}(\boldsymbol{y})$ denote a competitive equilibrium given endowments $\boldsymbol{y}$.

- Let $\boldsymbol{\tau}$ denote a system of lump-sum transfers of endowments, $\sum_i \boldsymbol{\tau}_i = 0$.

- Second Welfare Theorem (under some regularity conditions).
  For any $\boldsymbol{c} \in \mathcal{P}$ there exists transfers $\boldsymbol{\tau}$ such that $(\boldsymbol{c}, \boldsymbol{p}) \in \mathcal{C}(\boldsymbol{y} + \boldsymbol{\tau})$.

- Main idea: any Pareto efficient allocation can be supported as a competitive equilibrium with an appropriate set of initial transfers.

- Remarks. Needs more than first welfare theorem.

  Key regularity conditions: *convex preferences* and the *cheaper-point* property — allocation $\boldsymbol{c}$ must leave every consumer with alternative choice that is in their budget set. Transfers can't be 'totally immiserating'.

Second Welfare Theorem: Discussion

- Roughly speaking, if you do not like the market outcome $\boldsymbol{c}^m$ and prefer (according to some criterion) some other Pareto efficient outcome $\boldsymbol{c}$, 'all' you need to do is implement the transfers $\boldsymbol{\tau}$ such that

$$(\boldsymbol{c}, \boldsymbol{p}) \in \mathcal{C}(\boldsymbol{y} + \boldsymbol{\tau})$$

  And then let the market work its magic. Easy.

- While perhaps naive as a theory of *economic policy*, the second welfare theorem does provide an algoritm for constructing a 'representative agent' with much weaker assumptions than Gorman aggregation.

- For this, we use the fact that any Pareto efficient allocation is the solution to a specific individual optimization problem, the planning problem.

- To give this more punch, let's also consider how it works with *risk*.

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Event Tree

- Time $t = 0, 1, 2, \ldots$
- Events $s_t \in S$, nodes in an event tree.
- Histories of events $s^t = (s_0\,,\, s_1\,,\, \ldots\,, s_t) = (s^{t-1}\,,\, s_t)$
- Unconditional probabilities of histories $\pi_t(s^t)$.

Exchange Economy

- Individuals $i = 1, 2, \ldots, I$.
- Individual endowments $\boldsymbol{y}_i$ with typical element $y_{it}(s^t) > 0$.
- Individual allocation $\boldsymbol{c}_i$ with typical element $c_{it}(s^t)$.
- An allocation $\boldsymbol{c} = \{\boldsymbol{c}_i\}$ is *feasible* if $c_{it}(s^t) \geq 0$ for every i, t and s^t and

$$\sum_i c_{it}(s^t) \leq \sum_i y_{it}(s^t), \qquad \text{for all } t, s^t$$

- Let $Y_t(s^t)$ denote the *aggregate endowment*

$$Y_t(s^t) \equiv \sum_i y_t^i(s^t)$$

- Prices $\boldsymbol{p}$ with typical element $p_t(s^t) \geq 0$.

Preferences

- Individuals rank outcomes using the *expected utility* criterion

$$U(\boldsymbol{c}_i) \equiv \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \, u(c_{it}(s^t)) \, \pi_t(s^t)$$

- Remarks.
  - heterogeneous endowments $\boldsymbol{y}_i$
  - but *identical utility function* $U(\boldsymbol{c}_i)$
  - time- and state-separable, same β, $u(c)$, and agree on $\pi_t(s^t)$
  - will see how to relax some of this

- To streamline exposition, assume $u'(c) > 0$, $u''(c) < 0$ and $u'(0) = +\infty$.

Alternative Trading Arrangements

- Arrow-Debreu (time-zero) approach:

  Single enormous market at time $t = 0$, in which there is trade in a *complete set of contingent claims* for all possible histories s^t.

  At subsequent periods, $t = 1, 2, \ldots,$ agreed-upon trades are carried out but *no further trading occurs*.

Alternative Trading Arrangements

- Radner (sequence of markets) approach:

  At each time $t = 0, 1, 2, \dots$ and history s^t there is a market in which there is trade in a *complete set of contingent claims* for all possible nodes $s^{t+1} = (s^t, s_{t+1})$ that immediately follow s^t.

  In other words, there is the possibility of *dynamic trading*, contingent on the realized history s^t.

- Remarks.

  – roughly, Arrow-Debreu time-zero approach has many more assets but many fewer trading dates than Radner sequence-of-markets approach

  – perhaps confusingly, the one-period-ahead contingent claims in the sequence-of-markets approach are known as *Arrow securities*

  – turns out that these two approaches yield identical allocations

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Pareto Problem

- Now consider how to find a Pareto efficient allocation.

- Planner chooses allocation $\boldsymbol{c} = \{\boldsymbol{c}_i\}$ to maximize

$$W(\boldsymbol{c}) = \sum_i \omega_i U(\boldsymbol{c}_i)$$

  where $\omega_i \geq 0$ are a set of *Pareto weights*, parameters of the problem.

- Planner chooses allocation subject to the sequence of resource constraints

$$\sum_i c_{it}(s^t) \leq \sum_i y_{it}(s^t) = Y_t(s^t), \qquad \text{for all } t, s^t$$

- A solution to this problem is Pareto efficient, no individual can be made better off without another being made worse off.

- Really a *family of problems* — by varying $\boldsymbol{\omega}$ we trace out the set of Pareto efficient allocations $\mathcal{P}$.

Pareto Problem

- Lagrangian with stochastic multiplier $\mu_t(s^t) \geq 0$ for each constraint

$$\mathcal{L} = \sum_i \omega_i \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u(c_{it}(s^t))\pi_t(s^t) + \sum_{t=0}^{\infty} \sum_{s^t} \mu_t(s^t) \sum_i \left[y_{it}(s^t) - c_{it}(s^t)\right]$$

- Lagrangian can be written more compactly as

$$\mathcal{L} = \sum_i \sum_{t=0}^{\infty} \sum_{s^t} \left\{\omega_i \beta^t u(c_{it}(s^t))\pi_t(s^t) + \mu_t(s^t)\big[y_{it}(s^t) - c_{it}(s^t)\big]\right\}$$

which reveals that, from the planner's point of view, this is really a sequence of static problems.

Pareto Problem

- First order conditions for $c_{it}(s^t)$ are

$$\omega_i \, \beta^t \, u'(c_{it}(s^t)) \, \pi_t(s^t) = \mu_t(s^t), \qquad \text{for all } i, t, s^t$$

- Hence taking the ratio of these for individual i and individual 1, say

$$\frac{\omega_i}{\omega_1} \frac{u'(c_{it}(s^t))}{u'(c_{1t}(s^t))} = 1$$

- Efficiency requires that MRS are equalized across individuals up to the time-invariant 'fixed effects' ω_i.

- Invert this to write $c_{it}(s^t)$ in terms of $c_{1t}(s^t)$, namely

$$c_{it}(s^t) = u'^{\,-1}\left(\frac{\omega_1}{\omega_i} \, u'(c_{1t}(s^t)) \right)$$

Pareto Problem

- Plug this into the resource constraint to get

$$\sum_i u'^{\,-1}\left(\frac{\omega_1}{\omega_i}\, u'(c_{1t}(s^t))\right) = Y_t(s^t), \qquad \text{for all } t, s^t$$

- This is a single nonlinear equation in $c_{1t}(s^t)$ that we can solve.

- Once we have found $c_{1t}(s^t)$ can then recover $c_{it}(s^t)$ for all other i.

Properties of the Solution

- Solutions given by time-invariant function of the form

$$c_{it} = g(\,\omega_i\,,\,Y_t\,;\,\boldsymbol{\omega}\,)$$

- Remarks.

  – *completely insured against idiosyncratic risk*, c_{it} does not depend on y_{it}

  – but exposed to *aggregate risk*, c_{it} depends on Y_t

  – insurance here is purely cross-sectional, as opposed to say intertemporal smoothing in a permanent income model, would also have the latter if planner could smooth Y_t, e.g., as in a production economy

  – only individual characteristic that matters is exogenous Pareto weight ω_i

  – *distribution-free*, cross-sectional distribution of endowments y_{it} realized at t does not matter, only aggregate Y_t matters

  – *history-free*, current Y_t is a sufficient statistic for whole history

Constant Relative Risk Aversion

- EXAMPLE. Suppose the canonical CRRA utility function

$$u(c) = \frac{c^{1-\theta} - 1}{1-\theta}, \qquad \theta > 0$$

- Solution has the specific form

$$c_{it} = \left(\frac{\omega_i^{1/\theta}}{\sum_i \omega_i^{1/\theta}} \right) Y_t$$

- REMARKS.

  – each individual gets a fixed, time-invariant, share of Y_t with the size of that share increasing in their welfare weight ω_i

  – all *time series* properties of c_{it} driven by aggregate Y_t.

  – all *cross section* properties of c_{it} driven by ω_i, less dispersion when risk aversion θ is higher, for example

$$\text{std}[\log c_{it}] = \frac{1}{\theta}\text{std}[\log \omega_i]$$

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Arrow-Debreu (Time-Zero) Problem

- Now let's see the decentralized counterpart.

- Let $p_t(s^t)$ denote the price *at date $t = 0$* of a claim to one unit of consumption for delivery at t, s^t.

- Taking the prices as given, individuals choose $\boldsymbol{c}_i$ to maximize

$$U(\boldsymbol{c}_i) \equiv \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \, u(c_{it}(s^t)) \, \pi_t(s^t)$$

  subject to the single intertemporal budget constraint

$$\sum_{t=0}^{\infty} \sum_{s^t} p_t(s^t) c_{it}(s^t) \leq \sum_{t=0}^{\infty} \sum_{s^t} p_t(s^t) y_{it}(s^t)$$

- RHS is the time-zero value of their future endowments, i.e., their *initial wealth*. The LHS is the time-zero value of consumption.

Equilibrium

- A *competitive equilibrium* is a feasible allocation $\boldsymbol{c}$ and prices $\boldsymbol{p}$ such that

  (i) taking $\boldsymbol{p}$ as given, $\boldsymbol{c}_i$ is optimal for each i

  (ii) markets clear

$$\sum_i c_{it}(s^t) = \sum_i y_{it}(s^t), \qquad \text{for all } t, s^t$$

Arrow-Debreu Problem

- Lagrangian with *single multiplier* $\lambda_i \geq 0$ on budget constraint

$$\mathcal{L} = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u(c_{it}(s^t))\pi_t(s^t) + \lambda_i \sum_{t=0}^{\infty} \sum_{s^t} p_t(s^t)\big[y_{it}(s^t) - c_{it}(s^t)\big]$$

- Again, this can be written more compactly as

$$\mathcal{L} = \sum_{t=0}^{\infty} \sum_{s^t} \big\{\beta^t u(c_{it}(s^t))\pi_t(s^t) + \lambda_i \, p_t(s^t)\big[y_{it}(s^t) - c_{it}(s^t)\big]\big\}$$

Arrow-Debreu Problem

- First order conditions for $c_{it}(s^t)$ are

$$\beta^t\, u'(c_{it}(s^t))\, \pi_t(s^t) = \lambda_i\, p_t(s^t), \qquad \text{for all } t, s^t$$

- Hence taking the ratio of these for individual i and individual 1, say

$$\frac{u'(c_{it}(s^t))}{u'(c_{1t}(s^t))} = \frac{\lambda_i}{\lambda_1}$$

- Clearly the same *risk-sharing* condition as the planner if $\lambda_i = 1/\omega_i$.

- Invert this to write $c_{it}(s^t)$ in terms of $c_t^1(s^t)$, namely

$$c_{it}(s^t) = u'^{\,-1}\left(\frac{\lambda_i}{\lambda_1}\, u'(c_{1t}(s^t))\right)$$

Arrow-Debreu Problem

- For this to be an equilibrium allocation it must be feasible

$$\sum_i u'^{\,-1}\left(\frac{\lambda_i}{\lambda_1}\,u'(c_{1t}(s^t))\right) = Y_t(s^t)$$

- This is a single nonlinear equation in $c_t^1(s^t)$ that we can solve.

- Once we have found $c_{1t}(s^t)$ can then recover $c_{it}(s^t)$ for all other i. Gives

$$c_{it} = f(\lambda_i\,,\, Y_t\,; \boldsymbol{\lambda})$$

- Again complete insurance against idiosyncratic risk, history matters only through realization of aggregate endowment Y_t etc.

- But this is *not* a solution to the general equilibrium problem. Unlike the weights $\boldsymbol{\omega}$ in the planner's problem, the multipliers $\boldsymbol{\lambda}$ are endogenous.

Arrow-Debreu Problem

- If this was an individual decision problem, we would look to solve for their λ_i in terms of market prices $\boldsymbol{p}$ and their individual endowments $\boldsymbol{y}_i$.

- But this is a general equilibrium problem, there is feedback from $\boldsymbol{\lambda}$ to $\boldsymbol{p}$.

- Evaluate the intertemporal budget constraint of individual i at consumption $c_{it}(s^t) = f(\lambda_i, Y_t\,;\,\boldsymbol{\lambda})$ and prices

$$p_t(s^t) = \beta^t \, \frac{u'(f(\lambda_1, Y_t\,;\,\boldsymbol{\lambda}))}{\lambda_1} \, \pi_t(s^t)$$

  to get, for each $i = 1, 2, \ldots$

$$\sum_{t=0}^{\infty} \sum_{s^t} \beta^t \, u'(f(\lambda_1, Y_t\,;\,\boldsymbol{\lambda})) \, \pi_t(s^t) \, [f(\lambda_i, Y_t\,;\,\boldsymbol{\lambda}) - y_{it}(s^t)] = 0$$

- A system of nonlinear equations in unknown $\boldsymbol{\lambda}$. Homogeneous of degree zero in $\boldsymbol{\lambda}$ (always enter as λ_i/λ_1), so can choose a normalization.

Equilibrium and Planning Allocations

- If it turns out that $\lambda_i = 1/\omega_i$ then the equilibrium allocation coincides with the planning allocation (corresponding to $\boldsymbol{\omega}$).

- Put differently, there is a set of planning solutions indexed by the configuration of $\boldsymbol{\omega}$ and the competitive equilibrium *picks out one particular* solution, the one for which the planner has $\omega_i = 1/\lambda_i$.

- Since λ_i is inversely related to individual i's wealth, i.e., the market value of endowments, this is equivalent to saying the competitive equilibrium picks out the solution for which the planner *gives high weight to wealthy individuals and low weight to poor individuals*.

- At these weights, the planner's multipliers, i.e., shadow prices, $\mu_t(s^t)$, coincide with the equilibrium prices $p_t(s^t)$.

- As in the second welfare theorem, we can induce different equilibrium allocations by an appropriate choice of initial transfers $\boldsymbol{\tau}$.

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Constant Relative Risk Aversion

- Example. Suppose again CRRA utility with coefficient $\theta > 0$.

- In the competitive equilibrium, individuals have consumption

$$c_{it} = \left(\frac{\lambda_i^{-1/\theta}}{\sum_i \lambda_i^{-1/\theta}}\right) Y_t \equiv \alpha_i Y_t$$

  where the consumption shares α_i map 1-to-1 to λ_i and sum to 1.

- Corresponds to planner's solution if $\lambda_i = 1/\omega_i$.

- Up to a normalization, prices are then

$$p_t(s^t) = \beta^t \, Y_t^{-\theta} \, \pi_t(s^t)$$

- Prices reflect marginal utility of 'representative agent' with endowment Y_t.

Constant Relative Risk Aversion

- Can then solve for consumption share α_i using the budget constraints

$$\alpha_i = \frac{\displaystyle\sum_{t=0}^{\infty}\sum_{s^t} \beta^t \, \frac{y_{it}(s^t)}{Y_t(s^t)} \, Y_t^{1-\theta}(s^t)\pi_t(s^t)}{\displaystyle\sum_{t=0}^{\infty}\sum_{s^t} \beta^t Y_t^{1-\theta}(s^t)\pi_t(s^t)}$$

- Consumption share α_i is a weighted average of endowment shares

$$\alpha_i = \mathbb{E}_{\Omega}\left\{ \frac{y_{it}}{Y_t} \right\}$$

with weights

$$\Omega_t(s^t) \equiv \frac{\beta^t \, Y_t^{1-\theta}(s^t)\pi_t(s^t)}{\displaystyle\sum_{t=0}^{\infty}\sum_{s^t} \beta^t Y_t^{1-\theta}(s^t)\pi_t(s^t)}$$

Heterogeneity in Discount Factors, β_i

- Example. Suppose again CRRA utility with coefficient $\theta > 0$.

- Now suppose discount factors β_i and for convenience order them

$$1 > \beta_1 > \beta_2 > \cdots > 0$$

- Focus on planning problem with weights $\boldsymbol{\omega}$. Planner's solution

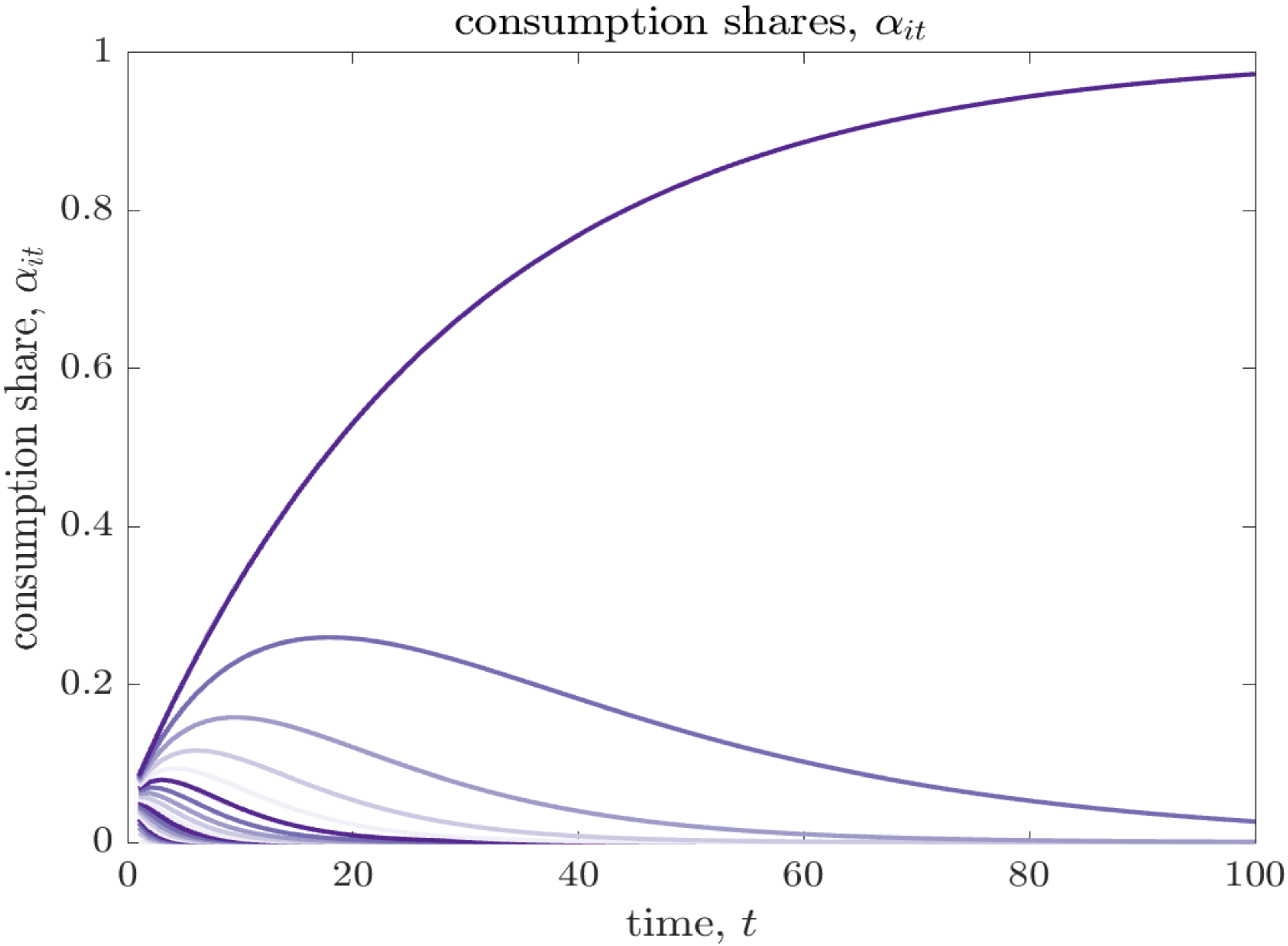
$$c_{it}(s^t) = \Big(\frac{\omega_i^{1/\theta}\beta_i^{t/\theta}}{\sum_i \omega_i^{1/\theta}\beta_i^{t/\theta}}\Big) Y_t(s^t) \equiv \alpha_{it} Y_t$$

- Time-varying shares α_{it} with deterministic dynamics. Relative shares

$$\Delta \log\big(\frac{\alpha_{it}}{\alpha_{1t}}\big) = \frac{1}{\theta}\log\big(\frac{\beta_i}{\beta_1}\big) < 0$$

- Asymptotically $\alpha_{1t} \to 1$ for most patient individual, $\alpha_{it} \to 0$ for all others.

Heterogeneity in Discount Factors, β_i



Outline

Heterogeneity in β_i and θ_i

- EXAMPLE. Suppose individual utility function is

$$U_i = \sum_{t=0}^{\infty} \sum_{s^t} \beta_i^t \, \frac{c_{it}(s^t)^{1-\theta_i}}{1-\theta_i} \, \pi_t(s^t)$$

- What is the 'representative agent' for this problem?

- Can show representative agent has utility function

$$\tilde{U} = \sum_{t=0}^{\infty} \sum_{s^t} \beta_1^t \, \tilde{u}(C_t(s^t); \tilde{\boldsymbol{\omega}}_t) \, \pi_t(s^t)$$

where $\tilde{\boldsymbol{\omega}}_t$ denote the time-varying 'adjusted' Pareto weights

$$\tilde{\omega}_{it} \equiv \frac{\omega_i}{\omega_1} \left(\frac{\beta_i}{\beta_1}\right)^t$$

and where the period utility function is the solution to the static problem

$$\tilde{u}(C\,;\,\tilde{\boldsymbol{\omega}}) \equiv \max_{\boldsymbol{c}_i} \Big[\sum_i \tilde{\omega}_i \frac{c_i^{1-\theta_i}}{1-\theta_i} \quad \text{subject to} \quad \sum_i c_i = C \Big]$$

Discussion

- Preferences not *identically* homothetic, do not satisfy Gorman form.

- Can still construct representative agent using second welfare theorem.

- In what sense is this representative agent *representative* of individual i?

  – discount factor $\beta_1 = \max \beta_i$

  – time- and state-dependent relative risk aversion

$$\tilde{\theta}_t = -\frac{\tilde{u}''(C_t; \tilde{\boldsymbol{\omega}}_t) C_t}{\tilde{u}'(C_t; \tilde{\boldsymbol{\omega}}_t)} > 0$$

- This kind of representative agent seems less suitable for welfare analysis.

Taking Stock

- Neoclassical growth model leans heavily on representative agent construct.

- Implicitly, complete risk-sharing to eliminate idiosyncratic risk and strong assumptions on preferences to justify normative interpretations.

- Since early 1990s, large literature on macro with *incomplete markets* where idiosyncratic risk cannot be eliminated (Imrohoroglu 1989, Huggett 1993, Aiyagari 1994, Krusell-Smith 1998 etc).

- In particular, Aiyagari (1994) is neoclassical growth model with idiosyncratic risk and incomplete markets. Kicked into overdrive a now massive literature on micro heterogeneity meets macro.

Next Class

- Overlapping generations.

- Possibility of *dynamic inefficiency*.

- Implications for capital accumulation and public debt.

Homework

- Consider a static planning problem

$$\tilde{u}(C;\boldsymbol{\omega}) \equiv \max_{\boldsymbol{c}_i} \Big[\sum_i \omega_i u_i(c_i) \quad \text{subject to} \quad \sum_i c_i = C \Big]$$

- Suppose two individuals, $i = 1, 2$, with constant *absolute* risk aversion (CARA) utility with coefficients γ_i, namely

$$u_i(c_i) = -\frac{\exp(-\gamma_i c_i)}{\gamma_i}, \qquad \gamma_i > 0$$

- Check. Show that the representative agent also has CARA utility

$$\tilde{u}(C;\boldsymbol{\omega}) = -A(\boldsymbol{\omega})\frac{\exp(-\tilde{\gamma} C)}{\tilde{\gamma}}, \qquad \tilde{\gamma} = \frac{\gamma_1\gamma_2}{\gamma_1 + \gamma_2} < \min[\gamma_1, \gamma_2]$$

Here the representative agent is *less risk averse than either* $i = 1, 2$.