Economic Growth

Lecture 6: Neoclassical growth model, part two

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Outline

1. **Comparative dynamics: worked examples**
   - Decrease in discount rate ρ.
   - Increase in productivity A
2. **Balanced Growth**
   - Preferences consistent with balanced growth
3. **Decentralization**
   - Equivalence to planning problem
   - Distortions: An example

Comparative Dynamics

- Recall basic neoclassical growth model in continuous time.

- System of two nonlinear differential equations in $c(t), k(t)$

$$\dot{c}(t) = \frac{Af'(k(t)) - \delta - \rho}{\theta}\, c(t)$$

$$\dot{k}(t) = Af(k(t)) - \delta k(t) - c(t), \qquad F(k, A) = Af(k)$$

- Boundary conditions, $k(0) > 0$ given and transversality condition.

- For convenience, suppose economy starts in steady state, $k(0) = k^*$.

- Consider *permanent*, *unanticipated* changes in parameters, determine path of $c(t), k(t)$ in response to such changes.

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Distortions: An example

Discount Rate $\rho_{\text{new}} < \rho$

- Consider permanent, unanticipated *decrease* in discount rate ρ.
- In short, an increase in the willingness to save.
- Work backwards from steady state effects to impact effects.
- Steady state effects:
  - steady state k^* determined by Euler equation

$$Af'(k^*) = \rho + \delta \qquad \Rightarrow \qquad \frac{dk^*}{d\rho} = \frac{1}{Af''(k^*)} < 0$$

    so a decrease in ρ increases steady state k^*

  - then steady state c^* determined by resource constraint

$$c^* = Af(k^*) - \delta k^* \qquad \Rightarrow \qquad \frac{dc^*}{d\rho} = \underbrace{[Af'(k^*) - \delta]}_{=\rho>0} \frac{dk^*}{d\rho} < 0$$

    so a decrease in ρ increases steady state c^* too

Discount Rate $\rho_{\mathbf{new}} < \rho$

- What about impact effects and subsequent dynamics? Redraw the phase diagram and see what's changed.

- No effect on $\dot{k}(t) = 0$ locus, $\mathcal{C}(k) \equiv Af(k) - \delta k$ is invariant to ρ.

- But shifts $\dot{c}(t) = 0$ locus, to the right in this case, since k^* decreasing in ρ.

- Stable arm shifts *down*, through new steady state c^{**}, k^{**}.

- IMPACT EFFECTS:

  – initial capital $k(0)$ is predetermined, here $k(0) = k^*$

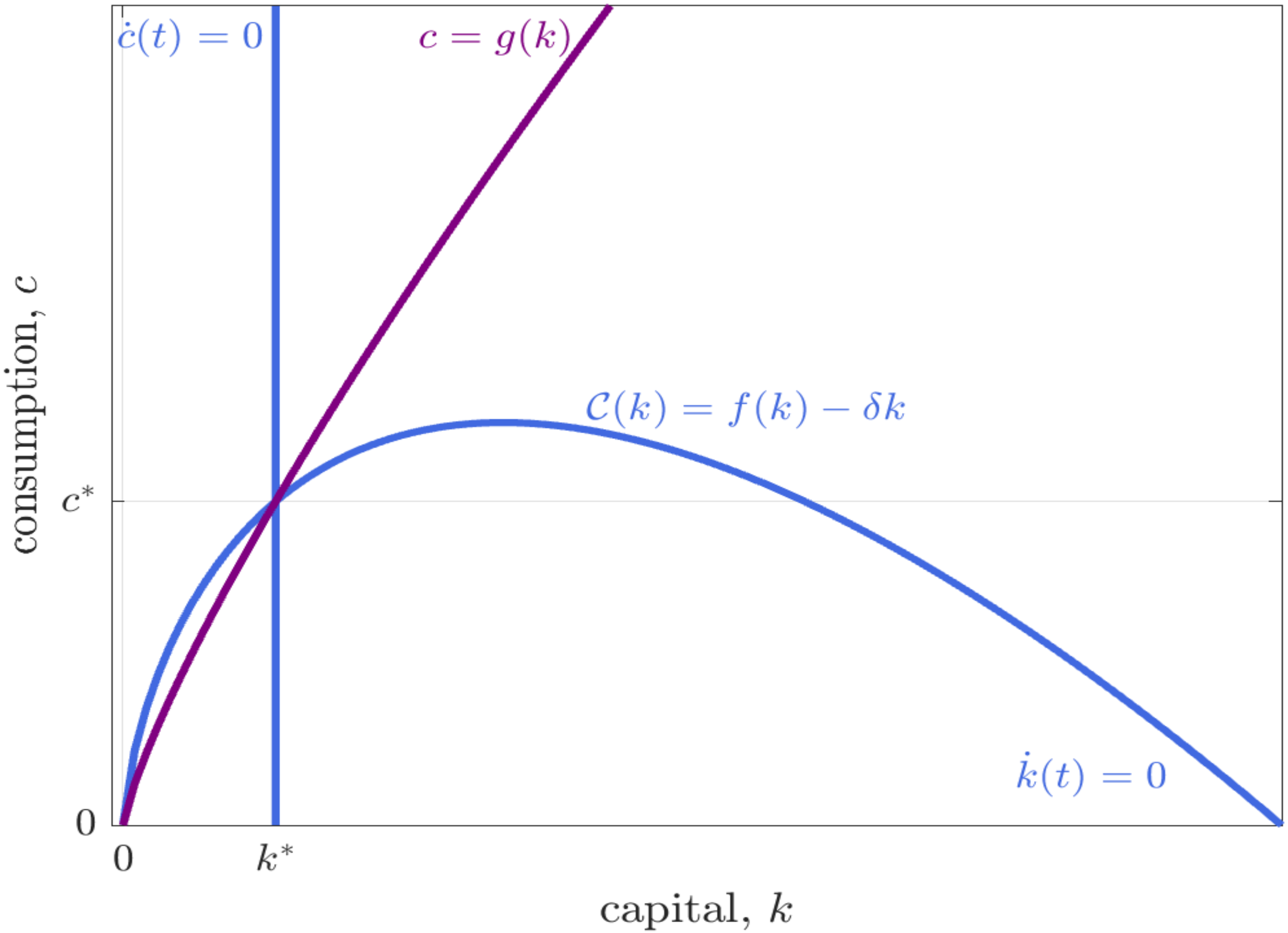
  – initial consumption $c(0)$ jumps to new stable arm through c^{**}, k^{**}

  – how do we know which way to jump?

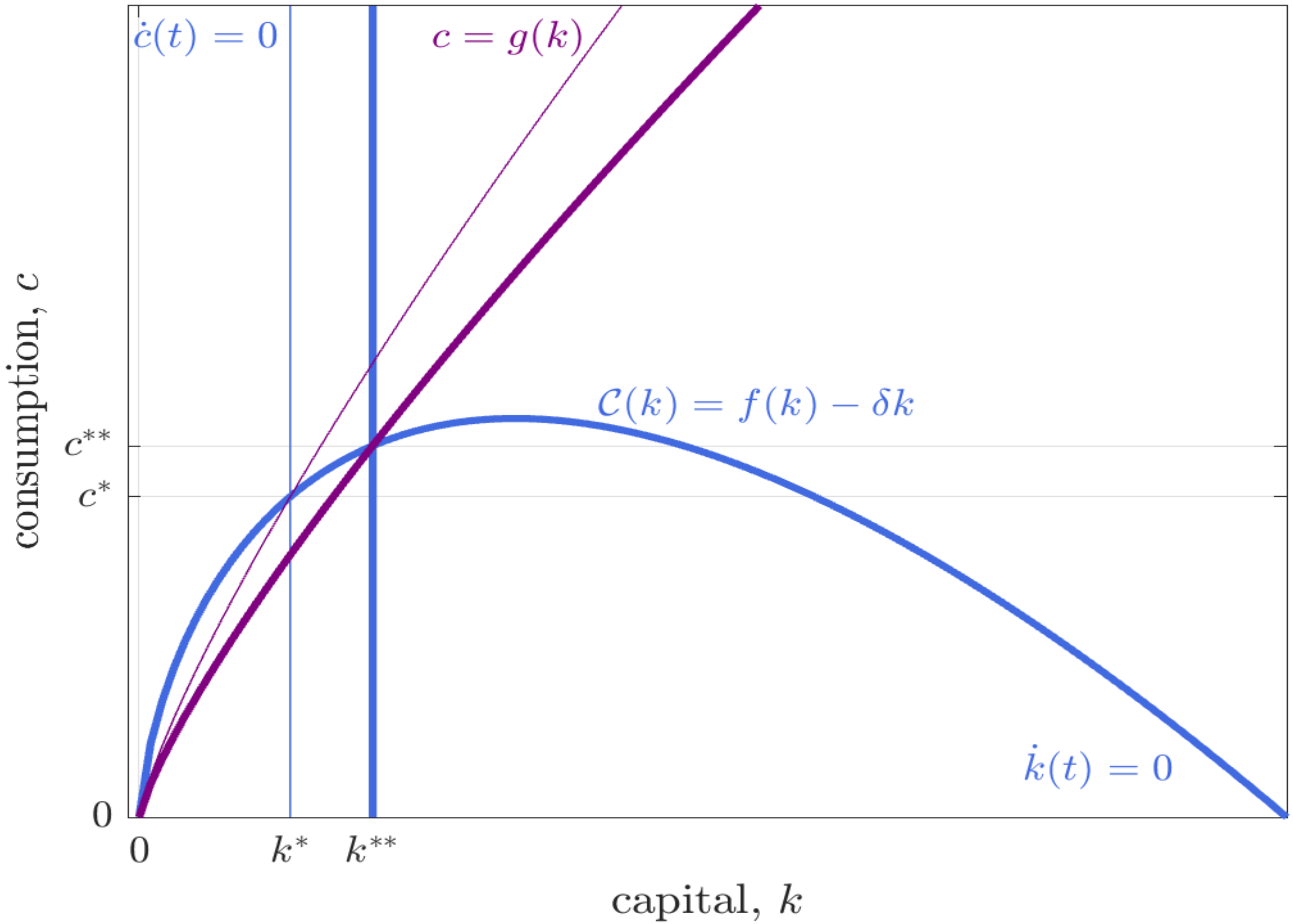
  – must jump down, to $c(0) < c^*$ to be on new stable arm

  – if jumped up, to $c(0) > c^*$, would be on diverging path

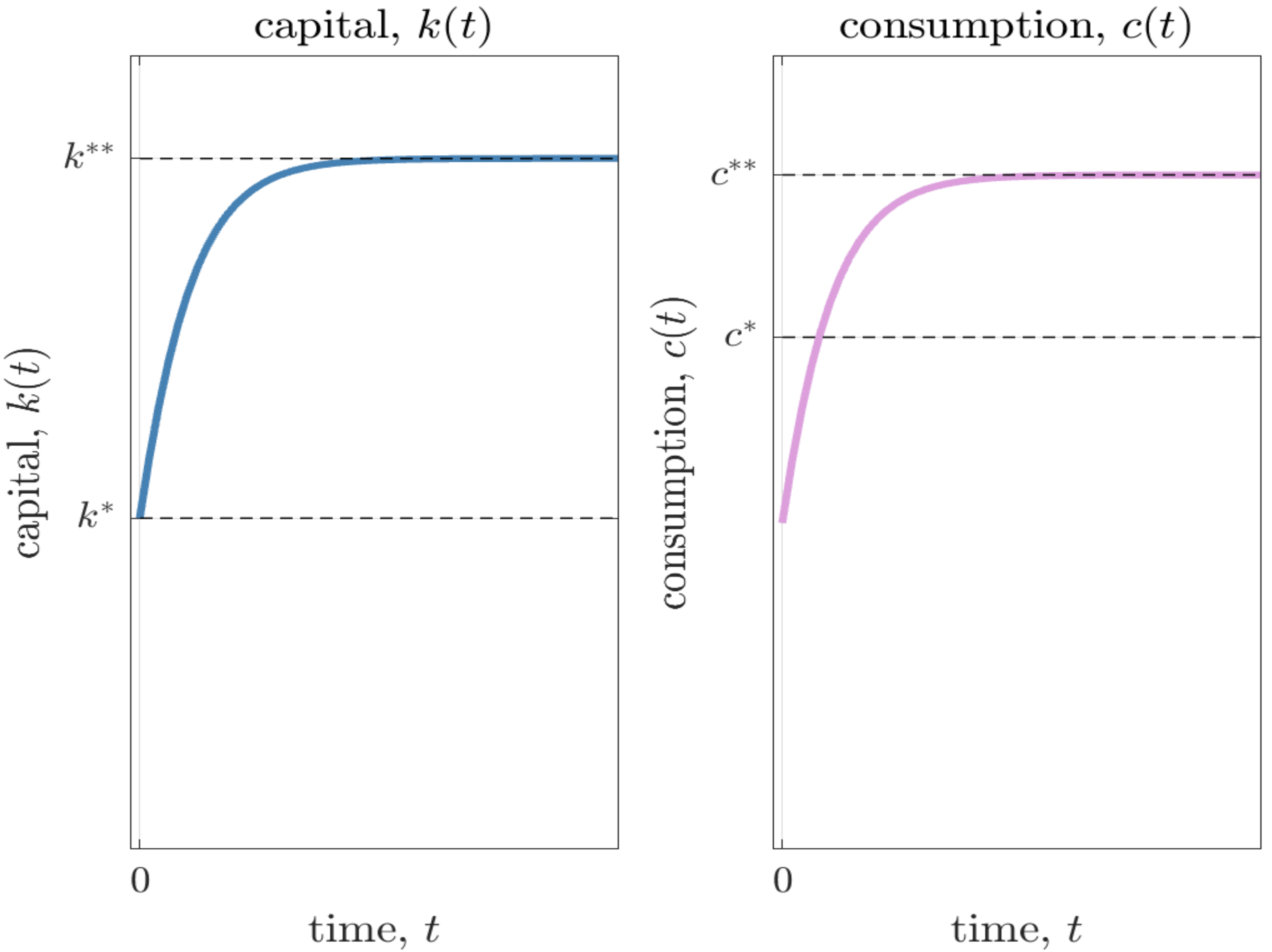
Discount Rate ρ



Discount Rate $\rho_{\text{new}} < \rho$



Discount Rate $\rho_{\text{new}} < \rho$



Discount Rate $\rho_{\mathbf{new}} < \rho$

- So, on impact, consumption falls, $c(0) < c^*$, then rises over time, passing its initial level c^* before converging to $c^{**} > c^*$.

- **Key intuition:** More patient, willing to have less consumption in short run to have more consumption in long run. On impact, resources unchanged but cut consumption, save, accumulate capital, and eventually have higher consumption.

- Notice that after the initial jump to $c(0) < c^*$, consumption is then always growing, consistent with the Euler equation

$$\frac{\dot{c}(t)}{c(t)} = \frac{Af'(k(t)) - \delta - \rho_{\text{new}}}{\theta} > 0, \qquad k(t) < k^{**}, \qquad t > 0$$

- Marginal product of capital does not change on impact, $Af'(k(0)) = \rho + \delta$, but is now high compared to new lower discount rate $\rho_{\text{new}} < \rho$.

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Productivity $A_{\text{new}} > A$

- Consider permanent, unanticipated *increase* in productivity A.
- Again work backwards from steady state effects to impact effects.
- STEADY STATE EFFECTS:
  - steady state k^* determined by Euler equation

$$Af'(k^*) = \rho + \delta \qquad \Rightarrow \qquad \frac{dk^*}{dA} = -\frac{f'(k^*)}{Af''(k^*)} > 0$$

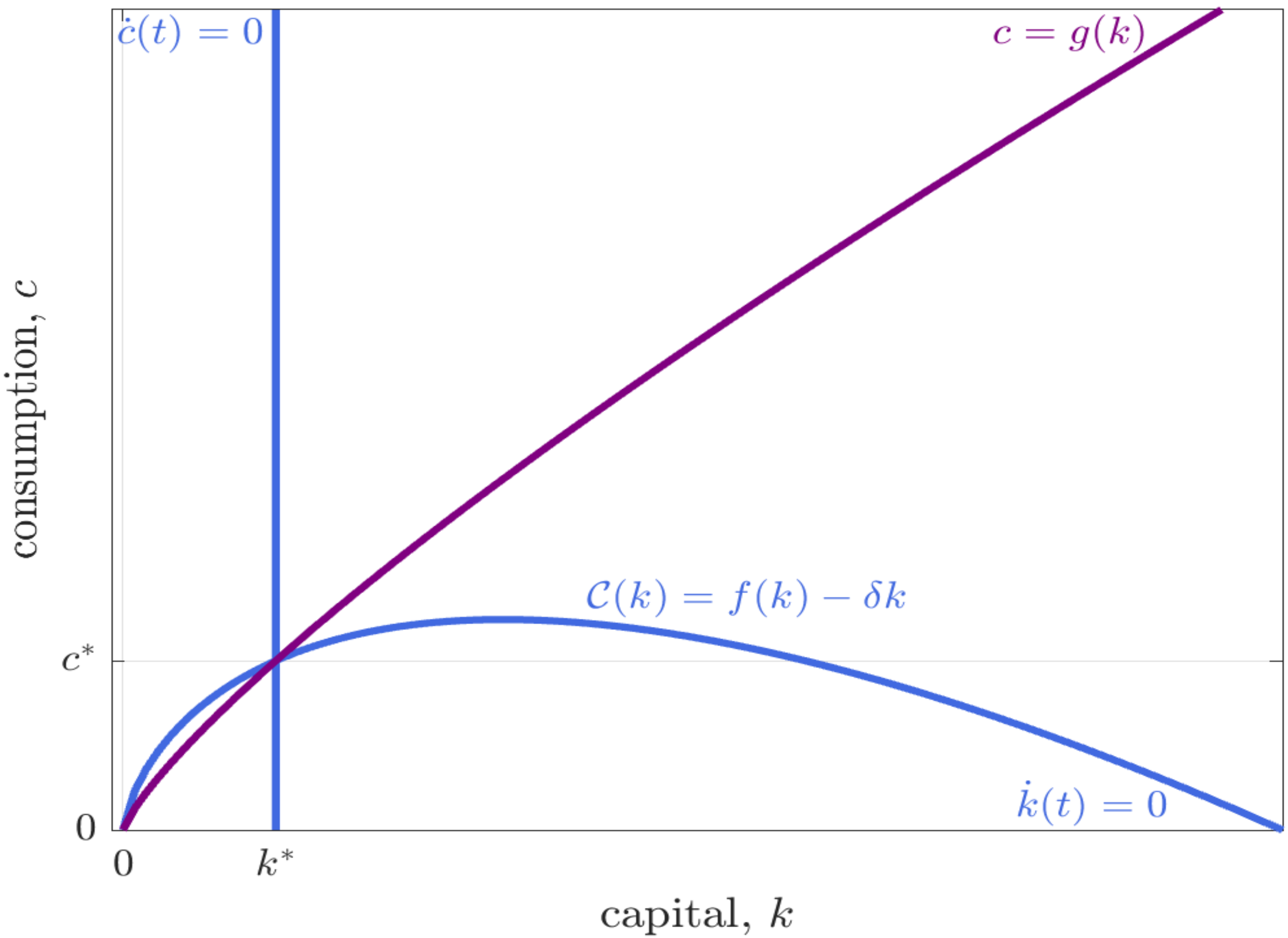
    so an increase in A increases steady state k^*

  - then steady state c^* determined by resource constraint

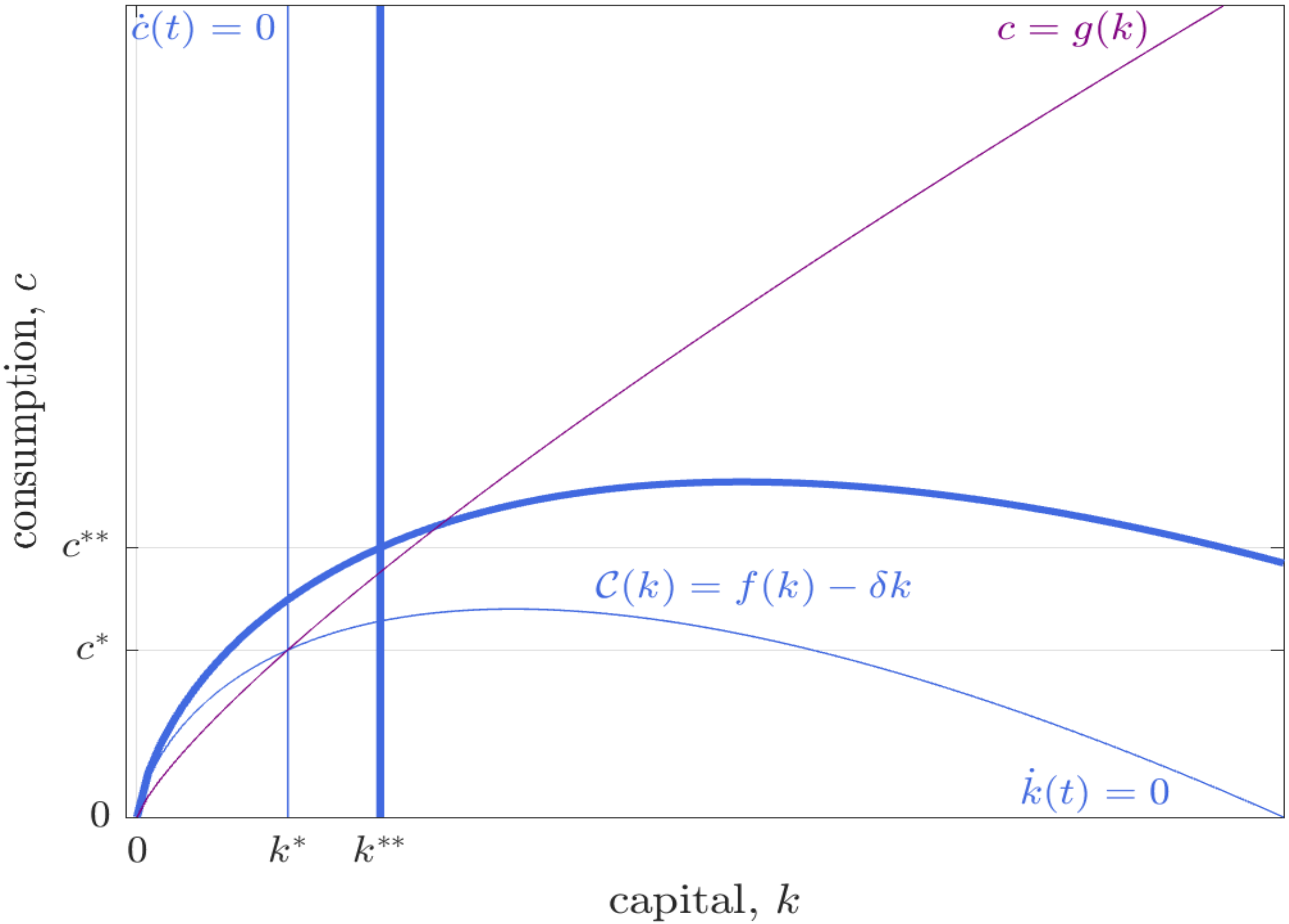
$$c^* = Af(k^*) - \delta k^* \qquad \Rightarrow \qquad \frac{dc^*}{dA} = f(k^*) + \underbrace{\left[Af'(k^*) - \delta\right]}_{=\rho>0} \frac{dk^*}{dA} > 0$$

    so an increase in A increases steady state c^* too

Productivity A



Productivity $A_{\text{new}} > A$



Productivity $A_{\mathbf{new}} > A$

- Long run effects seem superficially similar.

- But phase diagram is different in an important way.

- Increase in A shifts both $\dot{c}(t) = 0$ locus and $\dot{k}(t) = 0$ locus.

- Can no longer be sure that stable arm shifts down through new steady state c^{**}, k^{**}, stable arm *may* shift up.

- **Key intuition:** Intertemporal elasticity of substitution $\mathcal{E}(c) = 1/\theta$.

  – $\mathcal{E}(c)$ very high, very willing to substitute consumption over time, $c(0)$ jumps down on impact, $c(0) < c^*$, but then grows quickly to c^{**}

  – $\mathcal{E}(c)$ very low, very unwilling to substitute consumption over time, $c(0)$ jumps up on impact, $c(0) > c^*$ but then grows slowly to c^{**}

Productivity $A_{\text{new}} > A$

- To see this more formally, recall local dynamics governed by Jacobian

$$\boldsymbol{J} = \begin{pmatrix} 0 & \dfrac{Af''(k^*)}{\theta}c^* \\ & \\ -1 & \rho \end{pmatrix}$$

- Eigenvalues characterized by determinant and trace

$$\det(\boldsymbol{J}) = \lambda_1\lambda_2 = \frac{Af''(k^*)}{\theta}c^* < 0, \qquad \operatorname{tr}(\boldsymbol{J}) = \lambda_1 + \lambda_2 = \rho > 0$$

- Evaluate to

$$\lambda_1 = \frac{\rho - \sqrt{\rho^2 - 4\frac{Af''(k^*)c^*}{\theta}}}{2} < 0 < \frac{\rho + \sqrt{\rho^2 - 4\frac{Af''(k^*)c^*}{\theta}}}{2} = \lambda_2$$

- Stable root λ_1 gives speed of convergence, unstable root λ_2 gives slope of stable arm, i.e., policy function $c = g(k)$ has slope $g'(k^*) = \lambda_2$.

- Since $f''(k^*) < 0$, stable root λ_1 is increasing in θ while the unstable root λ_2 is decreasing in θ with $\lambda_1 \to 0^-$ and $\lambda_2 \to \rho^+$ as $\theta \to \infty$.

Productivity $A_{\text{new}} > A$

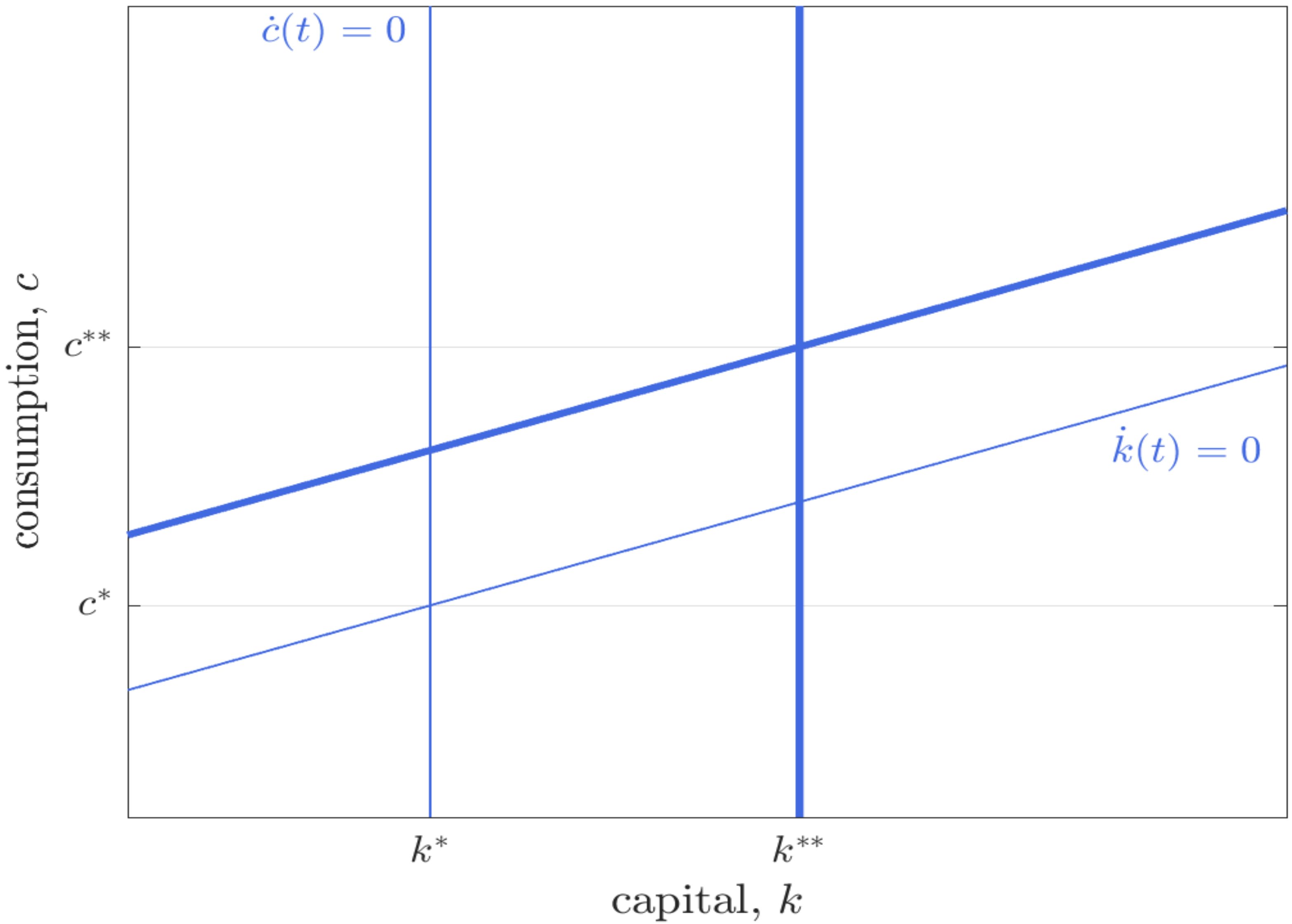
- *Perfect substitutes*
  - $\mathcal{E}(c) = 1/\theta$ high, i.e., θ low, eigenvalues both *large* in *magnitude*
  - speed of convergence is rapid and stable arm is steep
- *Perfect complements*
  - $\mathcal{E}(c) = 1/\theta$ low, i.e., θ high, eigenvalues both *small* in magnitude
  - speed of convergence is slow and stable arm is flat
- Example. Consider in particular $f(k) = k^{\alpha}$. Determinant evaluates to

$$\det(\boldsymbol{J}) = \lambda_1\lambda_2 = -\frac{1}{\theta} \times \left(\frac{(1-\alpha)(\rho+\delta)(\rho+(1-\alpha)\delta)}{\alpha}\right) < 0$$

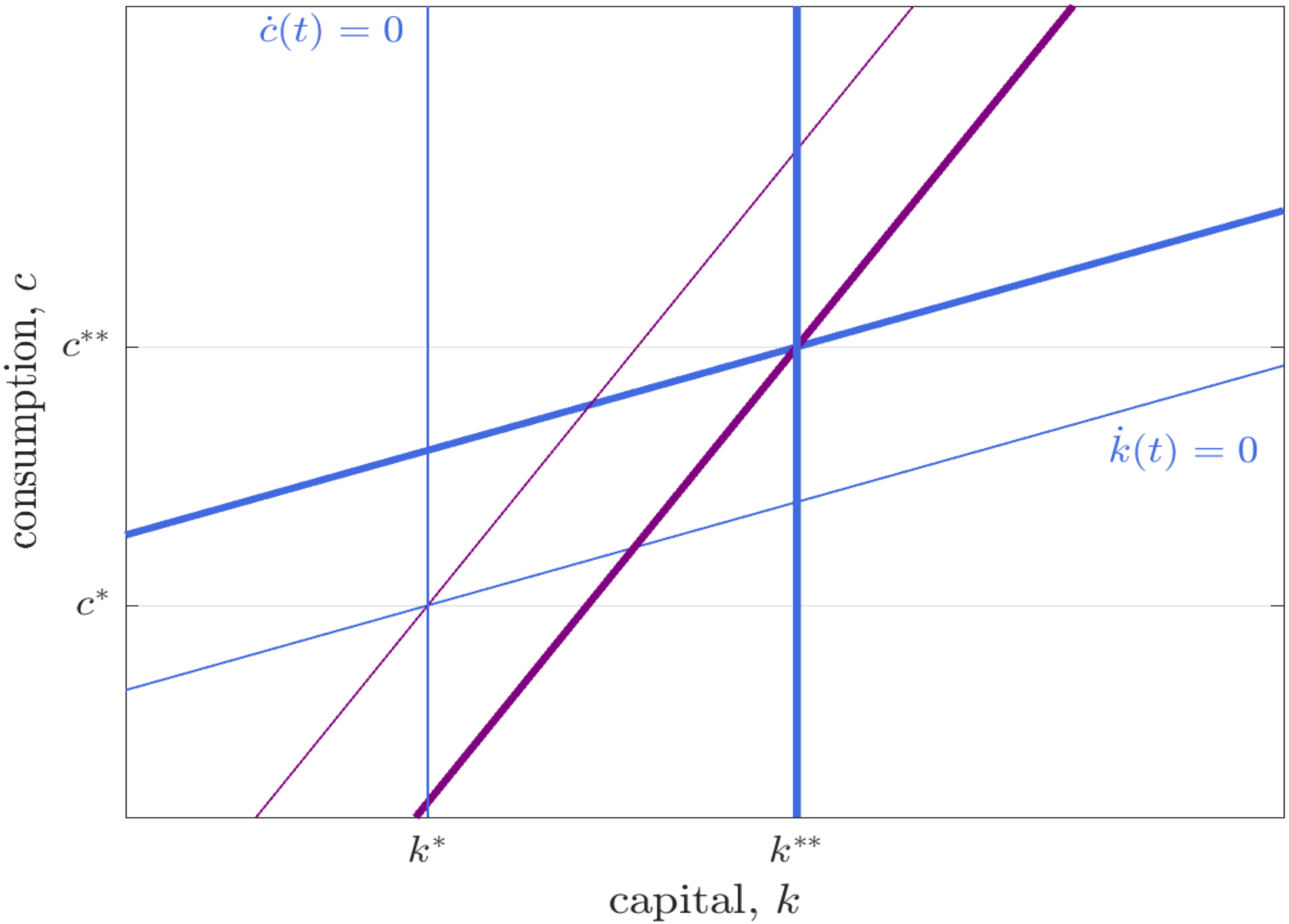
  which is *independent* of A, hence eigenvalues λ_1, λ_2 independent of A.

- Up to the local approximation, $A_{\text{new}} > A$ gives *parallel* shift in stable arm with slope governed by $\mathcal{E}(c) = 1/\theta$.

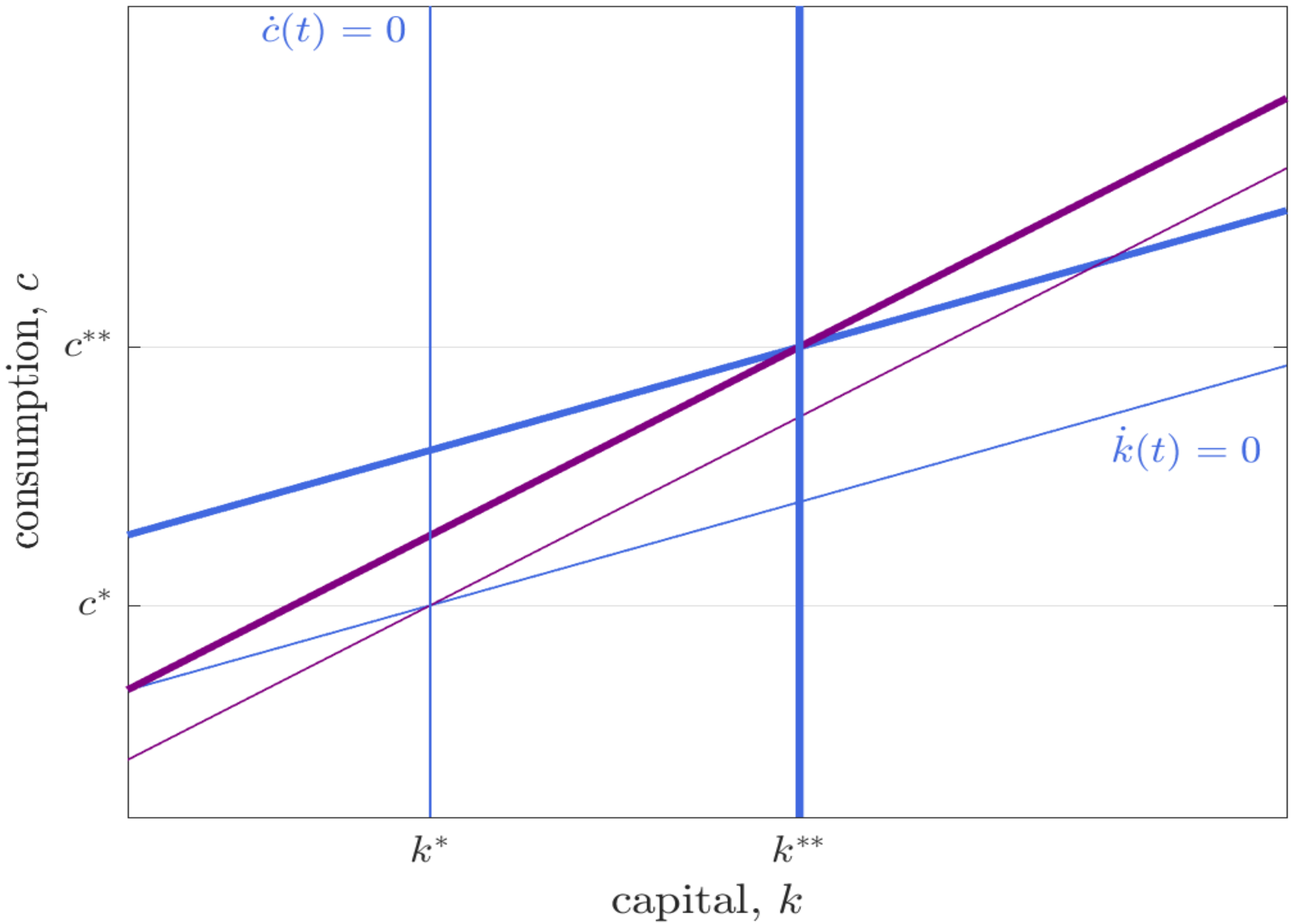
Productivity $A_{\text{new}} > A$: Local Dynamics



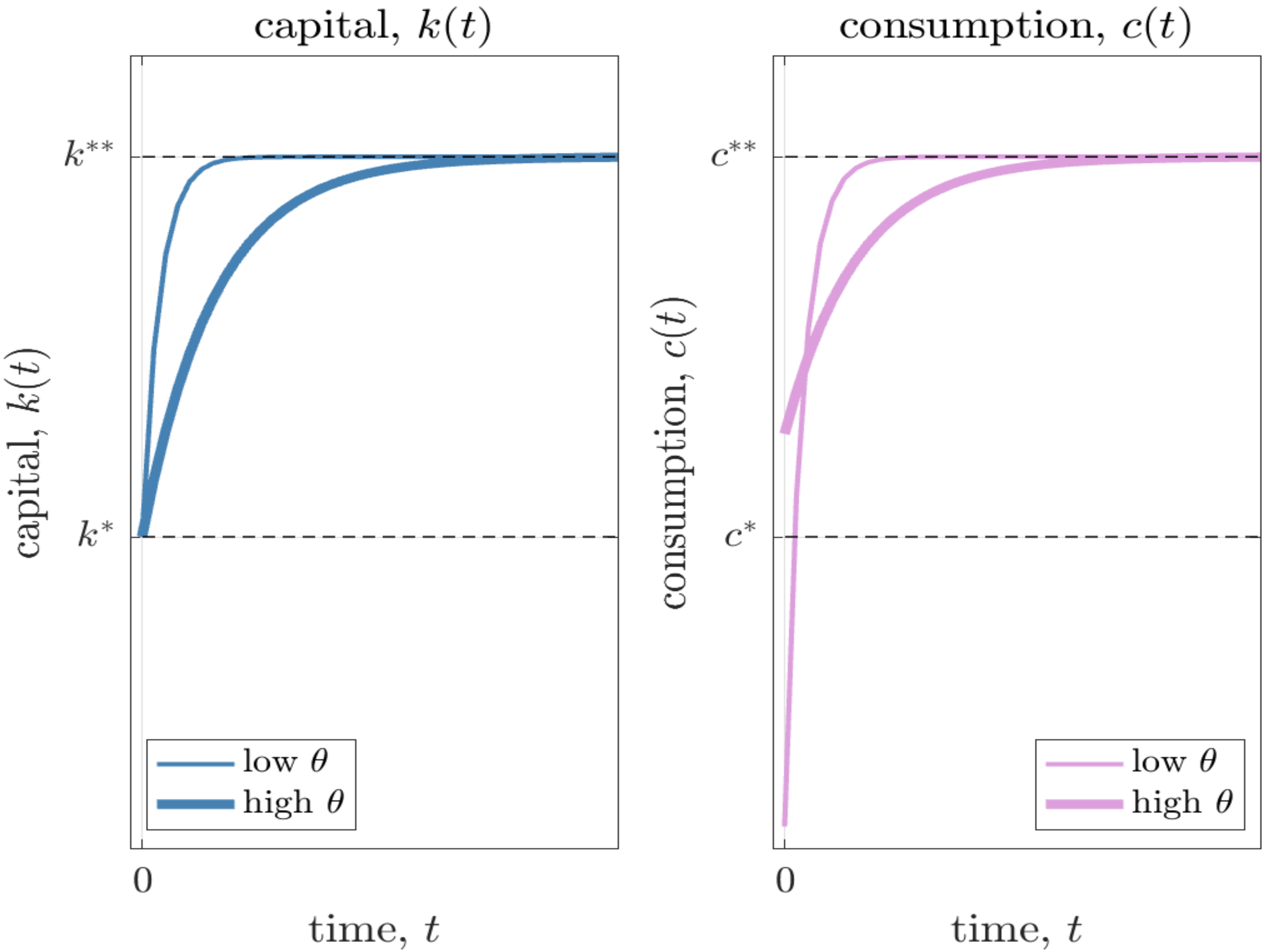
Productivity $A_{\text{new}} > A$: Low θ (High IES)



Productivity $A_{\mathrm{new}} > A$: High θ (Low IES)



Productivity $A_{\text{new}} > A$: Time Paths



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Distortions: An example

Adding Back Trend Growth

- Will now add back trend growth in A and L.

- Exogenous productivity and labor force growth

$$\dot{A}(t) = gA(t), \qquad \dot{L}(t) = nL(t)$$

- In anticipation of balanced growth, labor-augmenting productivity

$$Y(t) = F(K(t), A(t)L(t))$$

- Flow resource constraint

$$\dot{K}(t) = F(K(t), A(t)L(t)) - \delta K(t) - C(t), \qquad K(0) > 0$$

Planning Problem

- Planner seeks to maximize utility of $L(0) > 0$ identical households

$$U = \int_0^\infty e^{-\rho t}\, u\Big(\frac{C(t)}{L(t)}\Big)\, L(t)\, dt \qquad \rho > 0$$

subject to the flow resource constraint

$$\dot{K}(t) = F(K(t), A(t)L(t)) - \delta K(t) - C(t), \qquad K(0) > 0$$

and exogenous productivity and labor

$$A(t) = e^{gt} A(0), \qquad L(t) = e^{nt} L(0)$$

- What restrictions does balanced growth place on *preferences*? What assumptions do we need to ensure this problem is well-defined?

- Write consumption per effective worker, capital per effective worker etc

$$c(t) \equiv \frac{C(t)}{A(t)L(t)}, \qquad k(t) \equiv \frac{K(t)}{A(t)L(t)}$$

Planning Problem

- So growth in capital per effective worker is

$$\frac{\dot{k}(t)}{k(t)} = \frac{\dot{K}(t)}{K(t)} - (g+n)$$

$$= \frac{F(K(t), A(t)L(t)) - C(t)}{K(t)} - (\delta + g + n)$$

$$= \frac{f(k(t)) - c(t)}{k(t)} - (\delta + g + n)$$

- Hence resource constraint in intensive form is

$$\dot{k}(t) = f(k(t)) - (\delta + g + n)k(t) - c(t)$$

- With this in hand, write planning problem in $c(t), k(t)$.

Planning Problem

- Planner seeks to maximize utility of $L(0) > 0$ identical households

$$U = \int_0^\infty e^{-\rho t}\, u\big(c(t)A(t)\big)\, e^{nt} L(0)\, dt$$

subject to the flow resource constraint

$$\dot{k}(t) = f(k(t)) - (\delta + g + n)k(t) - c(t), \qquad k(0) > 0$$

- To streamline notation, normalize $A(0) = L(0) = 1$.

- Tentatively suggests we need at least:

- Assumption*. $\rho > n$.

- Remark. Is this strictly necessary? Or sufficient? Depends on growth in $u(t) \equiv u(c(t)A(t))$, as we will see.

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Planning Problem

- Proceed heuristically. Suppose the problem is well-defined.
- Write current-value Hamiltonian

$$\mathcal{H}(c, k, \lambda; A) \equiv u(cA) + \lambda(f(k) - (\delta + g + n)k - c)$$

- With effective discount rate $\rho - n > 0$, key optimality conditions

$$\mathcal{H}_c = 0, \qquad \mathcal{H}_k = (\rho - n)\lambda - \dot{\lambda}, \qquad \mathcal{H}_\lambda = \dot{k}$$

along with the transversality condition. Calculating the derivatives

$$\begin{aligned}
\mathcal{H}_c &= u'(cA)A - \lambda \\
\mathcal{H}_k &= \lambda(f'(k) - (\delta + g + n)) \\
\mathcal{H}_\lambda &= f(k) - (\delta + g + n)k - c
\end{aligned}$$

Planning Problem

- Collecting terms, our system of optimality conditions can be written

$$\lambda(t) = u'(c(t)A(t))A(t)$$
$$\dot{\lambda}(t) = (\rho - f'(k(t)) + (\delta + g))\lambda(t)$$
$$\dot{k}(t) = f(k(t)) - (\delta + g + n)k(t) - c(t)$$

along with the transversality condition and given initial condition.

- As usual, try to reduce this to a system in $c(t), k(t)$ by eliminating $\lambda(t)$.

- Take log derivatives of the first condition

$$\frac{\dot{\lambda}(t)}{\lambda(t)} = \frac{\dot{A}(t)}{A(t)} + \frac{u''(c(t)A(t))}{u'(c(t)A(t))}\left(\dot{c}(t)A(t) + c(t)\dot{A}(t)\right)$$

$$= g - \frac{1}{\mathcal{E}(c(t)A(t))} \times \left(\frac{\dot{c}(t)}{c(t)} + g\right), \qquad \mathcal{E}(x) \equiv -\frac{u'(x)}{u''(x)x}$$

where $\mathcal{E}(x)$ is the intertemporal elasticity of substitution at x.

Planning Problem

- Eliminating the multipliers and simplifying gives

$$\frac{\dot{c}(t)}{c(t)} + g = \mathcal{E}(c(t)A(t)) \times (f'(k(t)) - \delta - \rho)$$

- The left hand side is the growth rate of consumption per worker, $C(t)/L(t) = c(t)A(t)$, so this is just the usual consumption Euler equation.

- Now tentatively consider a balanced growth path where $c(t) \to c^*$ and $k(t) \to k^*$ for some c^*, k^* to be determined.

- For this to occur, it has to be the case that

$$g = \mathcal{E}(c^* A(t)) \times (f'(k^*) - \delta - \rho))$$

- For this to occur, it has to be the case that the intertemporal elasticity of substitution $\mathcal{E}(c^* A(t))$ is constant even as $A(t)$ grows exponentially.

- In short, at least *asymptotically*, $\mathcal{E}(x)$ must be constant. Incentives to substitute consumption over time independent of *level* of consumption.

Balanced Growth Preferences

- So, at least *asymptotically*, $\mathcal{E}(x)$ must have the constant elasticity form

$$u(x) = \frac{x^{1-\theta} - 1}{1-\theta}, \qquad \mathcal{E}(x) \equiv -\frac{u'(x)}{u''(x)x} = \frac{1}{\theta} > 0$$

- The the consumption Euler equation with trend growth simplifies to

$$\frac{\dot{c}(t)}{c(t)} = \frac{f'(k(t)) - \delta - \rho - \theta g}{\theta}$$

- Essentially the same as in the basic model, but with $\rho + \theta g$ instead of ρ.

- Now consider the planner's objective with these preferences.

Back to the Planner's Objective

- With this class of preferences, the planner's objective is

$$U = \int_0^\infty e^{-(\rho-n)t} \, \frac{(c(t)A(t))^{1-\theta}}{1-\theta} \, dt = \int_0^\infty e^{-(\rho-n)t} \, e^{(1-\theta)gt} \frac{c(t)^{1-\theta}}{1-\theta} \, dt$$

$$= \int_0^\infty e^{-(\rho-n-(1-\theta)g)t} \, \frac{c(t)^{1-\theta}}{1-\theta} \, dt$$

- So instead of ASSUMPTION* above, $\rho > n$, we really need:

- ASSUMPTION. $\rho > n + (1-\theta)g$.

- REMARKS. If $\theta \leq 1$, this is tighter than $\rho > n$. But weaker if $\theta > 1$ (usually considered the relevant case).

  Balanced growth puts restrictions on technology (labor-augmenting productivity) *and* preferences (constant intertemporal elasticity).

Dynamical System

- Let $\hat{\rho} \equiv \rho - n - (1-\theta)g > 0$ denote adjusted discount rate.

- Let $\hat{\delta} \equiv \delta + g + n$ denote adjusted depreciation rate.

- With these adjustments, dynamical system is then exactly as before

$$\dot{c}(t) = \frac{f'(k(t)) - \hat{\delta} - \hat{\rho}}{\theta} c(t)$$

$$\dot{k}(t) = f(k(t)) - \hat{\delta} k(t) - c(t)$$

- Two boundary conditions: (i) $k(0) > 0$ and (ii) transversality condition.

- Usual saddle path dynamics.

Transversality Condition

- Planner's transversality condition in per effective worker terms

$$\lim_{T\to\infty} e^{-\hat{\rho}T}u'(c(T))k(T) \leq 0$$

- For future reference, recall $u'(c(t)) = \lambda(t)$ and multipliers satisfy

$$\frac{\dot{\lambda}(t)}{\lambda(t)} = \hat{\rho} - f'(k(t)) + \hat{\delta}$$

- Integrating forward over $[0, T]$ we get

$$\lambda(T) = \lambda(0)\exp\left(\hat{\rho}T - \int_0^T (f'(k(s)) - \hat{\delta})\, ds\right)$$

- Since $\lambda(0) > 0$, the planner's transversality condition can be written

$$\lim_{T\to\infty} \exp\left(-\int_0^T (f'(k(s)) - \hat{\delta})\, ds\right)k(T) \leq 0$$

Long Run Outcomes

- Steady state given by

$$f'(k^*) = \hat{\rho} + \hat{\delta} = \rho + \theta g + \delta$$

and then

$$c^* = f(k^*) - \hat{\delta}k^* = f(k^*) - (\delta + g + n)k^*$$

- Balanced growth paths

$$\frac{K(t)}{L(t)} = k^* A(t)$$

and

$$\frac{Y(t)}{L(t)} = y^* A(t)$$

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Decentralization

- Let the final good be the numeraire.
- *Representative household*
  - many identical households
  - initial size $L(0) = 1$ growing at rate n
  - endowed with initial *net assets* $\mathcal{A}(0)$
  - takes wage rate $w(t)$ and real return on assets $r(t)$ as given
- *Representative firm*
  - identical technology $Y = F(K, AL)$
  - indeterminate number of firms, since constant returns to scale
  - take wage $w(t)$ and rental rate of capital $R(t)$ as given
  - choose labor and capital demand to max profits
  - owned by households, but constant returns to scale + competitive factor prices $\Rightarrow$ zero economic profits so nothing left to distribute to owners

Representative Household

- Consumption per worker $x(t) \equiv C(t)/L(t) = c(t)A(t)$.
- Intertemporal utility

$$U = \int_0^\infty e^{-\rho t}\, u(x(t))\, e^{nt}\, dt$$

- Standard interpretation: *dynastic preferences*, common discount rate ρ.
- Flow budget constraint in terms of net assets

$$\dot{\mathcal{A}}(t) = r(t)\mathcal{A}(t) + w(t)L(t) - C(t)$$

- Let $a(t) \equiv \mathcal{A}(t)/L(t)$ denote net assets per worker

$$\dot{a}(t) = (r(t) - n)a(t) + w(t) - x(t)$$

- Net assets can be negative, in which case household is *borrowing.*

Representative Household

- Could then have 'chain-letter' or 'Ponzi' scheme where household services existing debt by taking on new debt, driving consumption arbitrarily high.

- If such schemes were allowed, household does not face a meaningful budget constraint and optimization problem is degenerate.

- Can deal with this problem in several ways:

  (i) rule out borrowing, impose $a(t) \geq 0$
  (ii) allow borrowing up to some '*natural debt limit*'
  (iii) rule out Ponzi schemes directly

- Options (ii) and (iii) equivalent if natural debt limit correctly specified.

- We will focus on the latter.

No Ponzi Game Condition

- Let $p(t)$ denote the price of a unit of consumption at date t.

- In terms of the real return on assets $r(t)$ this is

$$p(t) = \exp\Big(-\int_0^t r(s)\,ds\Big) \qquad \Leftrightarrow \qquad \frac{\dot{p}(t)}{p(t)} = -r(t)$$

  (e.g., if $r(t) = r$ all t then $p(t) = e^{-rt}$)

- No Ponzi game condition is that, asymptotically, household must have net assets with non-negative present value

$$\lim_{T\to\infty} p(T)\mathcal{A}(T) \geq 0$$

  or equivalently, in terms of net assets per worker

$$\lim_{T\to\infty} \exp\Big(-\int_0^T (r(s)-n)\,ds\Big)a(T) \geq 0$$

Aside: Intertemporal Budget Constraint

- Consider a *finite* horizon $t \in [0, T]$.

- Write flow budget constraint in terms of $p(t)$

$$\underbrace{\dot{p}(t)\mathcal{A}(t) + p(t)\dot{\mathcal{A}}(t)}_{=\frac{d}{dt}p(t)\mathcal{A}(t)} = p(t)(w(t)L(t) - C(t))$$

- Integrating forward over $[0, T]$ we get

$$p(T)\mathcal{A}(T) + \int_0^T p(t)C(t)\,dt = \mathcal{A}(0) + \int_0^T p(t)w(t)L(t)\,dt$$

  (where implicitly $p(0) = 1$)

- Taking the limit $T \to \infty$ and imposing the no Ponzi game condition we get the *intertemporal budget constraint*

$$\int_0^\infty p(t)C(t)\,dt = \mathcal{A}(0) + \int_0^\infty p(t)w(t)L(t)\,dt$$

Representative Household

- Putting all this together, the household's problem is to maximize

$$U = \int_0^\infty e^{-(\rho-n)t}\, u(x(t))\, dt$$

subject to the flow budget constraint, in per worker terms

$$\dot{a}(t) = (r(t)-n)a(t) + w(t) - x(t)$$

and the no Ponzi game condition

$$\lim_{T\to\infty} \exp\Big(-\int_0^T (r(s)-n)\, ds\Big)a(T) \geq 0$$

which together imply the intertemporal budget constraint.

- Current-value Hamiltonian for this problem

$$\mathcal{H}(x,a,\lambda) = u(x) + \lambda((r-n)a + w - x)$$

Representative Household

- Current-value Hamiltonian for this problem

$$\mathcal{H}(x, a, \lambda) = u(x) + \lambda((r - n)a + w - x)$$

- Key optimality conditions

$$\mathcal{H}_x = 0, \qquad \mathcal{H}_a = (\rho - n)\lambda - \dot{\lambda}, \qquad \mathcal{H}_\lambda = \dot{a}$$

along with the transversality condition

$$\lim_{T\to\infty} e^{-(\rho-n)T}\lambda(T)a(T) = 0$$

- Calculating the derivatives of the Hamiltonian

$$\begin{aligned}
\mathcal{H}_x &= u'(x) - \lambda \\
\mathcal{H}_a &= \lambda(r - n) \\
\mathcal{H}_\lambda &= (r - n)a + w - x
\end{aligned}$$

Representative Household

- Eliminating the multipliers $\lambda(t)$ in the usual way we get

$$\frac{\dot{x}(t)}{x(t)} = \mathcal{E}(x(t)) \times (r(t) - \rho)$$

- With the usual constant elasticity specification $\mathcal{E}(x) = 1/\theta$ and writing consumption per worker $x(t) = c(t)A(t)$ this simplifies to

$$\frac{\dot{c}(t)}{c(t)} = \frac{r(t) - \rho - \theta g}{\theta}$$

analogous to the corresponding condition for the planner.

- Notice that the household's multipliers satisfy

$$\lambda(t) = \lambda(0) \exp\Big(- \int_0^t (r(s) - \rho)\, ds\Big)$$

We can use this to examine more closely the connection between the no Ponzi condition and the household's transversality condition.

NPG vs. TVC

- No Ponzi condition

$$\lim_{T\to\infty} \exp\left(-\int_0^T (r(s)-n)\,ds\right)a(T) \geq 0 \tag{NPG}$$

- Transversality condition, having substituted out $\lambda(t)$, can be written

$$\lim_{T\to\infty} \exp\left(-\int_0^T (r(s)-n)\,ds\right)a(T) \leq 0 \tag{TVC}$$

- The former is a primitive, part of the household's budget constraint.
- The latter is one of the household's *optimality* conditions.
- Notice that if the TVC is satisfied as an equality, the NPG is satisfied too.
- The planner has a TVC but does not face an NPG because the planner faces *resource constraints*, not *budget constraints*.

Assets and Capital Market

- In principle, households can hold portfolio of physical capital and bonds.
- But no risk. All assets *perfect substitutes.* Degenerate portfolio problem.
- Market clearing will imply

$$K(t) = a(t)L(t)$$

- And by *no-arbitrage*, return on any asset actually held must be

$$r(t) = R(t) - \delta$$

where $R(t)$ is the rental rate for physical capital.

Representative Firm

- Taking w and R as given choose L and K to max profits

$$F(K, AL) - wL - RK$$

- In terms of capital per effective worker, $k \equiv K/AL$, factor prices satisfy

$$R = f'(k)$$

and

$$w = (f(k) - f'(k)k)A$$

Competitive Equilibrium

- A *competitive equilibrium* is a system of prices and quantities such that:

  (i) the representative household maximizes utility, taking prices as given

  (ii) the representative firm maximizes profits, taking prices as given

  (iii) markets clear

- Finding a competitive equilibrium equivalent to solving planning problem.

Outline

1. Comparative dynamics: worked examples

Decrease in discount rate ρ.

Increase in productivity A

2. Balanced Growth

Preferences consistent with balanced growth

3. Decentralization

Equivalence to planning problem

Distortions: An example

Equivalence to Planning Problem

- Household consumption Euler equation

$$\frac{\dot{c}(t)}{c(t)} = \frac{r(t) - \rho - \theta g}{\theta}$$

- At $r(t) = R(t) - \delta$ and $R(t) = f'(k(t))$ this coincides with the planner's

$$\frac{\dot{c}(t)}{c(t)} = \frac{f'(k(t)) - \delta - \rho - \theta g}{\theta}$$

- Using growth adjusted parameters $\hat{\rho} \equiv \rho - n - (1-\theta) > 0$ and $\hat{\delta} \equiv \delta + g + n$ this is the same as

$$\frac{\dot{c}(t)}{c(t)} = \frac{f'(k(t)) - \hat{\delta} - \hat{\rho}}{\theta}$$

Equivalence with Planning Problem

- Household flow budget constraint

$$\dot{a}(t) = (r(t) - n)a(t) + w(t) - x(t)$$

- At $a(t) = K(t)/L(t) = k(t)A(t)$ and $x(t) = c(t)A(t)$, substituting out factor prices, $R(t) = f'(k(t))$ & $w(t) = (f(k(t)) - f'(k(t))k(t))A(t)$ gives

$$\begin{aligned}&\Big(\frac{\dot{k}(t)}{k(t)} + g\Big)k(t)A(t) \\ &= \Big(f'(k(t)) - \delta - n\Big)k(t)A(t) + (f(k(t)) - f'(k(t))k(t))A(t) - c(t)A(t)\end{aligned}$$

- Simplifying we get

$$\dot{k}(t) = f(k(t)) - \hat{\delta}k(t) - c(t), \qquad \hat{\delta} \equiv \delta + g + n$$

Coincides with planner's resource constraint in per effective worker terms.

Equivalence with Planning Problem

- Household transversality condition

$$\lim_{T\to\infty} \exp\Big(-\int_0^T (r(s)-n)\,ds\Big)a(T) = 0$$

- At $r(t) = f'(k(t)) - \delta$ and $a(t) = K(t)/L(t) = k(t)A(t)$ with $\hat{\delta} \equiv \delta + g + n$ this coincides with the planner's transversality condition

$$\lim_{T\to\infty} \exp\Big(-\int_0^T (f'(k(s)) - \hat{\delta})\,ds\Big)k(T) = 0$$

- In short, given an initial condition $k(0) > 0$, a path $c(t), k(t)$ that satisfies the planner's consumption Euler equation, resource constrain, and transversality condition is also a competitive equilibrium.

- Given quantities $c(t), k(t)$ recover *supporting* prices $w(t), r(t)$ and $p(t)$.

Implications for Prices

- On balanced growth path, rental rate and real interest rate constant

$$R^* = f'(k^*) = \hat{\rho} + \hat{\delta} = \rho + \theta g + \delta$$

and hence

$$r^* = f'(k^*) - \delta = \rho + \theta g$$

Key intuition: High θ (low intertemporal elasticity of substitution) implies household needs more compensation to accept growth g.

- So can also write consumption Euler equation

$$\frac{\dot{c}(t)}{c(t)} = \frac{r(t) - r^*}{\theta}$$

- On balanced growth path, wages grow with productivity

$$w(t) = (f(k^*) - f'(k^*)k^*)A(t)$$

Outline

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Distortions: An example

Distortions: An Example

- Consider a uniform distortionary tax τ on the use of capital with tax revenues rebated lump sum to representative household.

- Representative firm now hires capital until

$$(1+\tau)R = f'(k)$$

with wages

$$w = (f(k) - f'(k)k)A$$

- Representative household faces interest rate $r = R - \delta$. On balanced growth path $r = \rho + \theta g$. Steady state capital per effective worker given by

$$(1+\tau)(\rho + \theta g + \delta) = f'(k^*) \qquad \Rightarrow \qquad \frac{dk^*}{d\tau} = \frac{1}{1+\tau}\frac{f'(k^*)}{f''(k^*)} < 0$$

- Hence steady state output per effective worker

$$y^* = f(k^*) \qquad \Rightarrow \qquad \frac{dy^*}{d\tau} = \frac{1}{1+\tau}\frac{f'(k^*)^2}{f''(k^*)} < 0$$

Distortions: An Example

- Suppose $f(k) = k^\alpha$ and consider two countries identical except for τ. Long run relative capital per effective worker would then be

$$\frac{k_1^*}{k_2^*} = \left(\frac{1+\tau_2}{1+\tau_1}\right)^{\frac{1}{1-\alpha}}$$

- Long run relative output per effective worker would then be

$$\frac{y_1^*}{y_2^*} = \left(\frac{1+\tau_2}{1+\tau_1}\right)^{\frac{\alpha}{1-\alpha}}$$

- For a number like $\alpha = 1/3$, even a 9-fold difference in distortions would only lead to a 3-fold difference in relative output.

- Once again, strongly diminishing returns to capital limit the ability of the neoclassical growth model to generate large cross country variation.

  (But if we could believe $\alpha = 2/3$, then a 9-fold difference in distortions would lead to a much larger 81-fold difference in relative output...)

Next class

- Dynamic general equilibrium theory.

- Aggregation and the representative agent.

Homework

- For simplicity suppose constant A and $L = 1$. Suppose the representative consumer maximizes

$$U = \int_0^\infty e^{-\rho t}\, u(c(t))\, dt, \qquad u(c) = \frac{c^{1-\theta} - 1}{1 - \theta}$$

subject to the *intertemporal budget constraint*

$$\int_0^\infty p(t)c(t)\, dt = a(0) + \int_0^\infty p(t)w(t)\, dt$$

Set this problem up as a Lagrangian with single time-invariant multipler.

- Check. Use the Lagrangian to derive the consumption Euler equation

$$\frac{\dot{c}(t)}{c(t)} = \frac{r(t) - \rho}{\theta}, \qquad r(t) = -\frac{\dot{p}(t)}{p(t)}$$

(i.e., without the need to use a Hamiltonian)