Economic Growth

Lecture 5: Neoclassical growth model, part one

Chris Edmond

MIT 14.452

Fall 2021

Roadmap

- Recall the Solow model assumes an exogenous saving rate s.

- The *neoclassical growth model* studies *optimal* saving behavior.

  [Ramsey (1928), version here follows Cass (1965) and Koopmans (1965)]

- Next three lectures cover the neoclassical growth model and its connections to dynamic general equilibrium theory

(L5) intertemporal preferences, optimal saving in a basic production economy

(L6) further details, applications, examples, extensions

(L7) decentralization, dynamic general equilibrium theory

Roadmap

- Beyond its application to growth per se, the neoclassical growth model serves as a starting point for a vast array of applied macroeconomic models, not all of which have the same efficiency properties.

- Version here relies heavily on a *representative agent* assumption.

- We will discuss in detail what this assumption does and does not mean.

- In subsequent lectures we will turn to an *overlapping generations* model that dispenses with the representative agent assumption.

- But before any of this, a review of intertemporal choice.

Outline

1. **Intertemporal choice**

2. **Neoclassical growth model: planning problem**
   - Steady state
   - Transitional dynamics

3. **Neoclassical growth model: continuous time**

Choice

- Suppose consumption bundles $\boldsymbol{c} \geq 0$

- Preferences represented by concave utility function $U(\boldsymbol{c})$, *ranks* bundles

- Consumer faces prices $\boldsymbol{p}$ and has endowments $\boldsymbol{y}$, budget constraint

$$\boldsymbol{p} \cdot \boldsymbol{c} \leq \boldsymbol{p} \cdot \boldsymbol{y}$$

- Lagrangian with single multiplier $\lambda \geq 0$ is

$$U(\boldsymbol{c}) + \lambda \boldsymbol{p} \cdot (\boldsymbol{y} - \boldsymbol{c})$$

- System of first order necessary conditions

$$\nabla U(\boldsymbol{c}) = \lambda \boldsymbol{p}$$

Choice

- Solve first order conditions to get $\boldsymbol{c}(\lambda \boldsymbol{p})$
- Plug into budget constraint to pin down multipler

$$\boldsymbol{p} \cdot \boldsymbol{c}(\lambda \boldsymbol{p}) = \boldsymbol{p} \cdot \boldsymbol{y} \qquad \Rightarrow \qquad \lambda(\boldsymbol{p}, \boldsymbol{y})$$

- Solution is then

$$\boldsymbol{c}^*(\boldsymbol{p}, \boldsymbol{y}) = \boldsymbol{c}(\lambda(\boldsymbol{p}, \boldsymbol{y})\boldsymbol{p})$$

- Prices $\boldsymbol{p}$ matter both directly (substitution effects) and indirectly via multipler λ (income/wealth effects).

Intertemporal Choice

- Now let $\boldsymbol{c}$ represent the dated consumption stream

$$\boldsymbol{c} = \{c_0, \quad c_1, \quad c_2, \quad \ldots \quad \} = \{c_t\}$$

- And let $\boldsymbol{p}$ represent the prices of these dated consumption goods

$$\boldsymbol{p} = \{p_0, \quad p_1, \quad p_2, \quad \ldots \quad \} = \{p_t\}$$

- System of first order conditions $\nabla U(\boldsymbol{c}) = \lambda \boldsymbol{p}$ has typical element

$$U_{c,t}(\boldsymbol{c}) = \lambda p_t, \qquad t = 0, 1, 2, \ldots$$

  where $U_{c,t}(\boldsymbol{c})$ denotes the marginal utility of consumption at date t.

- Standard 'marginal rate of substitution (MRS) equals relative price' tangency condition

$$\frac{U_{c,t+1}(\boldsymbol{c})}{U_{c,t}(\boldsymbol{c})} = \frac{p_{t+1}}{p_t}$$

Time-Separable Utility

- We will typically specialize to the case of a *time-separable* utility function

$$U(\boldsymbol{c}) = \sum_{t=0}^{\infty} \beta^t u(c_t), \qquad 0 < \beta < 1$$

  with strictly concave period utility, $u'(c) > 0$, $u''(c) < 0$.

- Future utility is discounted by constant factor β

$$1, \quad \beta, \quad \beta^2, \quad \beta^3, \quad \ldots$$

- Marginal utility of date-t consumption

$$U_{c,t} = \beta^t u'(c_t)$$

  depends only on t and c_t, not consumption on any other date.

- Tangency condition simplifies to

$$\beta \frac{u'(c_{t+1})}{u'(c_t)} = \frac{p_{t+1}}{p_t}$$

Intertemporal Prices and Real Interest Rates

- Equivalently

$$-u'(c_t)\frac{1}{p_t}+\beta u'(c_{t+1})\frac{1}{p_{t+1}}=0$$

- Suppose consumer could trade in a riskless real bond, giving up one unit of consumption at t to get R_{t+1} units of consumption *for sure* at $t+1$.

- Consumer would be indifferent to holding such an investment if

$$-u'(c_t)+\beta u'(c_{t+1})R_{t+1}=0$$

- In short, intertemporal prices p_t and real interest rates are linked by

$$\frac{p_{t+1}}{p_t}=R_{t+1}^{-1} \qquad \Rightarrow \qquad \frac{p_t}{p_0}=\prod_{s=0}^{t}R_s^{-1}, \qquad t\geq 1$$

- Example. Suppose a constant R. Then $p_t/p_0=R^{-t}$.

Consumption Euler Equation

- This tangency condition is often written

$$u'(c_t) = \beta u'(c_{t+1})R_{t+1}, \qquad \text{or} \qquad \beta \frac{u'(c_{t+1})}{u'(c_t)} R_{t+1} = 1$$

- A standard optimality condition in consumption/savings problems, known as a *consumption Euler equation.*

- **Key intuition:** consuming 1 unit less at t costs $u'(c_t)$ utility but delivers R_{t+1} per unit at $t+1$ which when converted to utility and discounted back to t is a marginal benefit of $\beta u'(c_{t+1})R_{t+1}$.

  At the optimum, marginal cost equals marginal benefit.

- REMARK. Slight abuse of notation, this R is not in general the rental rate.

Consumption Dynamics

- Recall that $u''(c) < 0$. This implies

$$c_{t+1} > c_t \qquad \Leftrightarrow \qquad u'(c_{t+1}) < u'(c_t) \qquad \Leftrightarrow \qquad \beta R_{t+1} > 1$$

- Let $r = R - 1$ denote the *net* real interest rate and likewise let $\rho = 1/\beta - 1$ denote the discount rate [the '*pure rate of time preference*'].

- Then we have

$$c_{t+1} > c_t \qquad \Leftrightarrow \qquad r_{t+1} > \rho$$

- Qualitatively, consumption grows when real interest rate relatively high. Quantitatively, strength of this effect depends on attitudes to *intertemporal substitution*, embedded in curvature of $u'(c)$.

Intertemporal Substitution

- The *intertemporal elasticity of substitution* is the relative change in consumption in response to a change in relative prices along a given indifference curve [holding the level of utility constant]

$$\frac{d\log\left(\frac{c_{t+1}}{c_t}\right)}{d\log\left(\frac{p_{t+1}}{p_t}\right)}$$

- EXAMPLE. CES preferences

$$\tilde{U} = \left(\sum_{t=0}^{\infty} \beta^t c_t^{1-\theta}\right)^{\frac{1}{1-\theta}}, \qquad \theta > 0$$

  Implies constant intertemporal elasticity of substitution

$$\frac{d\log\left(\frac{c_{t+1}}{c_t}\right)}{d\log\left(\frac{p_{t+1}}{p_t}\right)} = -\frac{1}{\theta}$$

Intertemporal Substitution

- To see this, first observe that $\tilde{U}$ represents the same preferences as

$$U = \sum_{t=0}^{\infty} \beta^t u(c_t), \qquad u(c) = \frac{c^{1-\theta} - 1}{1-\theta}, \qquad \theta > 0$$

- For these preferences, the MRS equal relative price tangency condition is

$$\beta \frac{u'(c_{t+1})}{u'(c_t)} = \beta \left(\frac{c_{t+1}}{c_t} \right)^{-\theta} = \frac{p_{t+1}}{p_t}$$

or

$$\frac{c_{t+1}}{c_t} = \left(\beta \frac{p_t}{p_{t+1}} \right)^{1/\theta} = (\beta R_{t+1})^{1/\theta}$$

So indeed the elasticity is constant, $1/\theta$ in this parameterization.

Intertemporal Substitution

- Now recall the approximation $\log(1+x) \approx x$ for small x.

- Then $\rho \approx -\log\beta > 0$ and $r_{t+1} \approx \log R_{t+1} = -\log(p_{t+1}/p_t)$ so that

$$\log\left(\frac{c_{t+1}}{c_t}\right) \approx \frac{r_{t+1}-\rho}{\theta}$$

- In short, the intertemporal elasticity of substitution $1/\theta$ measures the sensitivity of consumption growth to real interest rates.

- Examples.

  - *perfect substitutes*, $\theta = 0$, consumption growth very sensitive to r_{t+1}

  - *perfect complements*, $\theta = \infty$, consumption growth invariant to r_{t+1}

  - *log utility*, $\theta = 1$, consumption growth responds 1-for-1 to r_{t+1}

Aside on Log Utility

- Special case $\theta = 1$ corresponds to *log utility*. Using l'Hôpital's rule

$$\lim_{\theta \to 1} \frac{c^{1-\theta} - 1}{1 - \theta} = \log c$$

- So intertemporal utility is

$$U = \sum_{t=0}^{\infty} \beta^t \log c_t$$

- This is a special case of Cobb-Douglas utility with 'weights' β^t.

- Has the familiar constant expenditure share property

$$\frac{p_{t+1} c_{t+1}}{p_t c_t} = \beta$$

Aside on Risk Aversion

- This is a deterministic model. There is no risk.

- But the CES parameter θ corresponds to the Arrow-Pratt *coefficient of relative risk aversion*

$$\mathcal{R}(c) \equiv -\frac{u''(c)c}{u'(c)} = \theta > 0$$

- Because of this, these CES preferences are sometimes known as CRRA [constant relative risk aversion] preferences.

- This functional form has the special property that attitudes to risk θ are bound up with attitudes to deterministic intertemporal substitution $1/\theta$, even though these are intrinsically distinct concepts.

Outline

1. **Intertemporal choice**

2. **Neoclassical growth model: planning problem**
   Steady state
   Transitional dynamics

3. **Neoclassical growth model: continuous time**

Neoclassical Growth Model: Planning Problem

- Discrete time $t = 0, 1, 2, \dots$
- To fix ideas, special case with $L_t = L$ and $A_t = A$.
- Aggregate production function

$$Y_t = F(K_t, AL)$$

- Goods may be either consumed or invested

$$C_t + I_t = Y_t$$

- Capital accumulation

$$K_{t+1} = (1-\delta)K_t + I_t, \qquad 0 < \delta < 1, \qquad K_0 > 0$$

- Gives the sequence of *resource constraints*, one for each date

$$C_t + K_{t+1} = F(K_t, AL) + (1-\delta)K_t, \qquad K_0 > 0$$

Neoclassical Growth Model: Planning Problem

- Planner seeks to maximize utility of L identical households

  $$\sum_{t=0}^{\infty} \beta^t \, u\Big(\frac{C_t}{L}\Big) \, L, \qquad 0 < \beta < 1$$

  subject to sequence of resource constraints, one for each date

  $$C_t + K_{t+1} = F(K_t, AL) + (1-\delta)K_t, \qquad K_0 > 0$$

- REMARKS. Usual interpretation is that this is a *benevolent* planner seeking to maximize the households' own (identical) utility function.

  Can construct a 'representative agent' in a broad range of settings, using the *second welfare theorem*, but then may lose some normative appeal.

  Infinite horizon keeps the model *stationary*, no life-cycle effects.

Neoclassical Growth Model: Planning Problem

- Let $c_t = C_t/L$ denote consumption per household, $k_t = K_t/L$ etc.
- Planner chooses stream $\boldsymbol{c}$ of consumption per household to maximize

$$U(\boldsymbol{c}) = \sum_{t=0}^{\infty} \beta^t u(c_t), \qquad 0 < \beta < 1$$

subject to sequence of resource constraints, one for each date

$$c_t + k_{t+1} = f(k_t) + (1-\delta)k_t, \qquad k_0 > 0$$

- Remarks. Intensive form of the production function $f(k) \equiv F(k, A)$.

  Other than the infinite horizon, this is a standard concave problem.

Neoclassical Growth Model: Planning Problem

- Lagrangian with multiplier $\lambda_t \geq 0$ for each resource constraint

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(c_t) + \sum_{t=0}^{\infty} \lambda_t \big[f(k_t) + (1-\delta)k_t - c_t - k_{t+1} \big]$$

- Key first order conditions, hold at each date

$$\begin{aligned} c_t : \quad & \beta^t u'(c_t) - \lambda_t = 0 \\ k_{t+1} : \quad & -\lambda_t + \lambda_{t+1}\big[f'(k_{t+1}) + 1 - \delta \big] = 0 \\ \lambda_t : \quad & f(k_t) + (1-\delta)k_t - c_t - k_{t+1} = 0 \end{aligned}$$

- Transversality condition, analogous to $k_{T+1} = 0$ in finite horizon problem

$$\lim_{T\to\infty} \lambda_T k_{T+1} = 0$$

Consumption Euler Equation Revisited

- Eliminating the multipliers, we get a consumption Euler equation

$$u'(c_t) = \beta u'(c_{t+1}) \big[\underbrace{f'(k_{t+1}) + 1 - \delta}_{\text{`}R_{t+1}\text{'}} \big]$$

- Marginal rate of transformation (MRT) between t and $t+1$

$$f'(k_{t+1}) + 1 - \delta$$

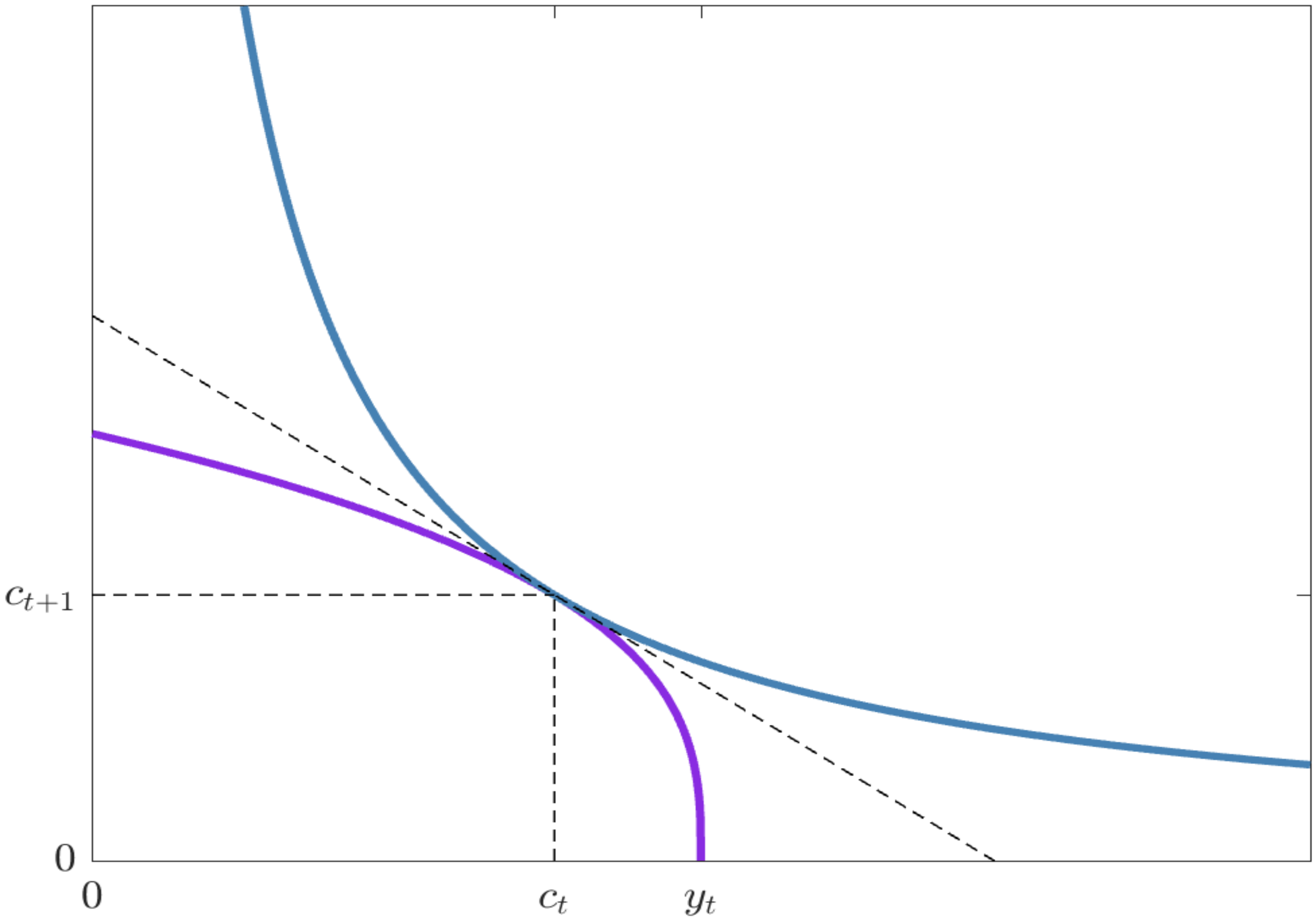
- Marginal rate of substitution (MRS) between t and $t+1$

$$\frac{u'(c_t)}{\beta u'(c_{t+1})}$$

- Planner equates MRS and MRT. Supported by *shadow prices*

$$\frac{\lambda_t}{\lambda_{t+1}}$$

Planner Equates Intertemporal MRS and MRT



Dynamical System

- Gives a system of two nonlinear difference equations in c_t, k_t

$$u'(c_t) = \beta u'(c_{t+1})\big[f'(k_{t+1}) + 1 - \delta\big]$$

  and

$$c_t + k_{t+1} = f(k_t) + (1-\delta)k_t$$

- Two boundary conditions: (i) $k_0 > 0$ and (ii) transversality condition

$$\lim_{T\to\infty} \beta^T u'(c_T)k_{T+1} = 0$$

Outline

Steady State

- Steady states are fixed points of the form $c_{t+1} = c_t$ and $k_{t+1} = k_t$.

- Let c^*, k^* denote steady state values. These are uniquely determined by

$$1 = \beta\big[f'(k^*) + 1 - \delta\big] \qquad \Leftrightarrow \qquad f'(k^*) = \rho + \delta$$

  and

$$c^* = f(k^*) - \delta k^*$$

- Steady state Euler equation first pins down k^*.

- Given steady state k^*, resource constraint then determines c^*.

Modified Golden Rule

- Let $\mathcal{C}(k)$ denote hypothetical steady state c given steady state k

$$\mathcal{C}(k) \equiv f(k) - \delta k$$

That is, $\{(c, k) : c = \mathcal{C}(k)\}$ is the set of points such that $k_{t+1} = k_t$.

- Note that $\mathcal{C}(k)$ is maximized at the '*golden rule*' level, where

$$f'(k^*_{\text{GR}}) = \delta$$

- Now have a '*modified golden rule*'. Steady state capital determined by

$$f'(k^*) = \rho + \delta$$

- Since $f''(k) < 0$ and $\rho > 0$, steady state capital is less than the golden rule level. Earlier consumption is preferred to later consumption.

Steady State Saving Rate

- Let s^* denote the steady state saving rate.

- Since savings equals investment, the steady state saving rate is

$$s^* = \frac{\delta k^*}{y^*} = \frac{\delta}{\rho + \delta} \frac{f'(k^*)k^*}{f(k^*)}$$

- Example. Suppose $f(k) = k^\alpha A^{1-\alpha}$. Recall in this case $s^*_{\text{GR}} = \alpha$ so

$$s^* = \frac{\delta}{\rho + \delta}\,\alpha = \frac{\delta}{\rho + \delta}\, s^*_{\text{GR}}$$

- Of course saving is in general time-varying outside steady state.

Outline

1. **Intertemporal choice**

2. **Neoclassical growth model: planning problem**
   - Steady state
   - Transitional dynamics

3. **Neoclassical growth model: continuous time**

Transitional Dynamics

- Consumption dynamics

$$c_{t+1} > c_t \qquad \Leftrightarrow \qquad k_{t+1} < k^*$$

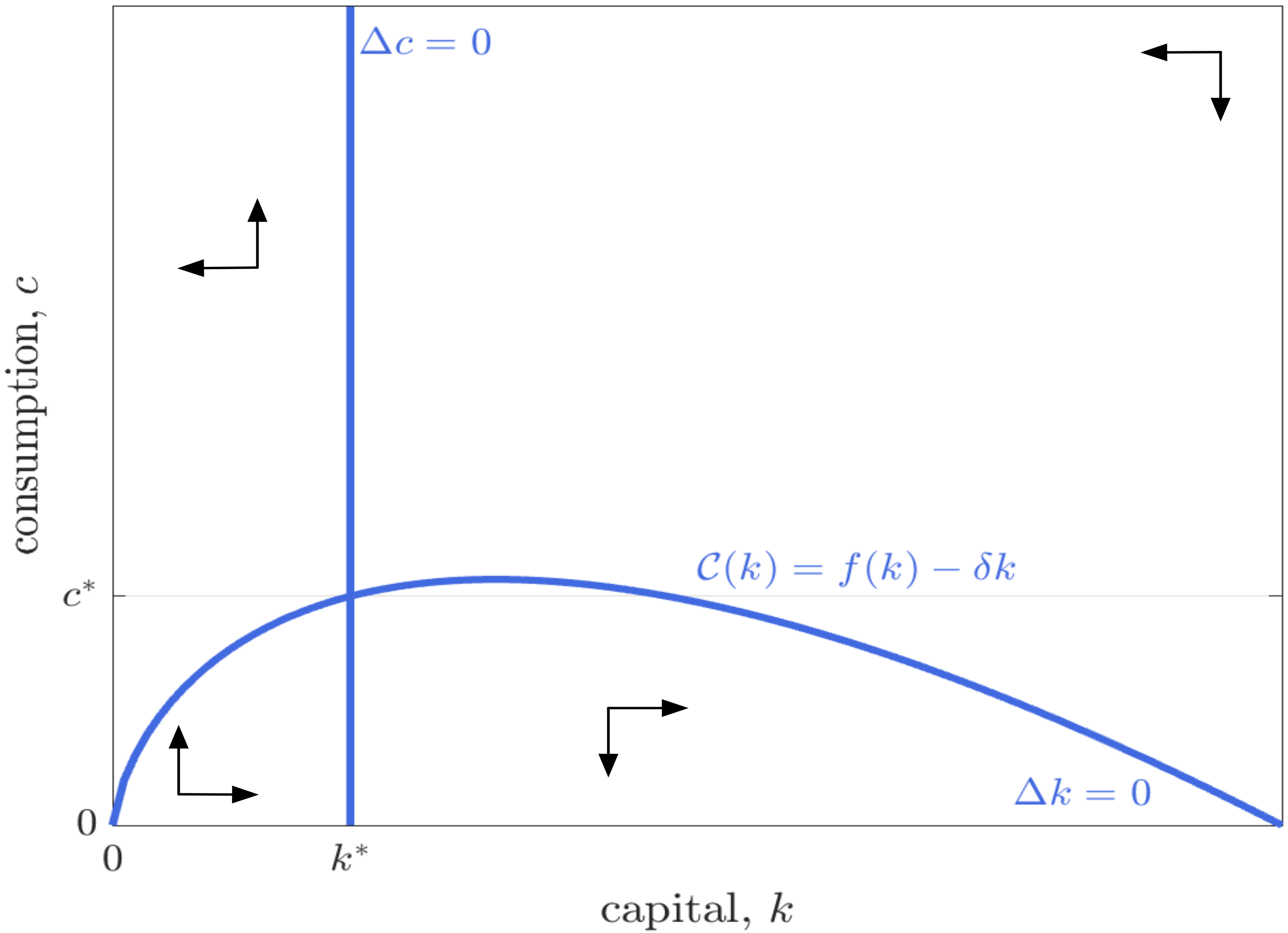
- Capital dynamics

$$k_{t+1} > k_t \qquad \Leftrightarrow \qquad c_t < \mathcal{C}(k_t)$$

- Partitions c_t, k_t space into *four regions*.

- Use phase diagram in c_t, k_t space to analyze dynamics.

Phase Diagram



Saddle Path

- Capital k_0 is *pre-determined* (historically given) at date $t = 0$

- Consumption c_0 not pre-determined, can take on any value in feasible set

$$0 \;\leq\; c_0 \;\leq\; f(k_0) + (1-\delta)k_0$$

- Dynamics *saddle-path unstable*. Almost all trajectories diverge from c^*, k^*.

- There is a single curve in c_t, k_t space that leads to the steady state, the *stable-arm*, which we can write

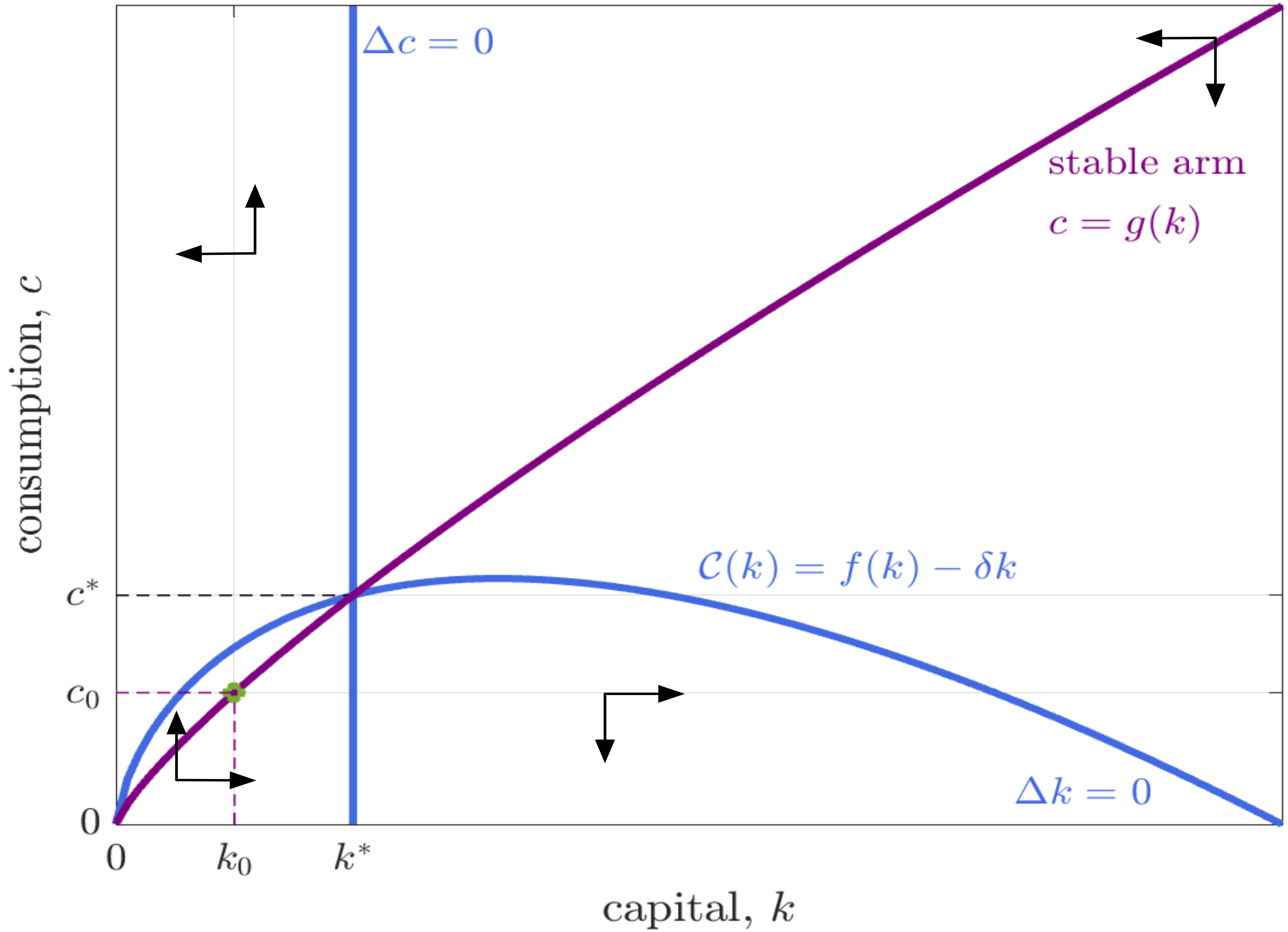
$$c_t = g(k_t)$$

  (in dynamic programming terms, this is the *policy function*)

- Initial consumption is the one degree of freedom that can be used to avoid diverging. Initial consumption *jumps* to the stable arm

$$c_0 = g(k_0)$$

Stable Arm



$\Delta c = 0$
stable arm
$c = g(k)$
$\mathcal{C}(k) = f(k) - \delta k$
$\Delta k = 0$
consumption, c
c^*
c_0
0
0
k_0
k^*
capital, k

Solving the Model

- Solving the model reduces to finding the function $g(k)$.
- Except in a handful of special cases, no closed form solutions.
- Powerful numerical methods exist to approximate $g(k)$ globally, but beyond the scope of this course.
- We will look at *local dynamics*, linear approximations around c^*, k^*.
- For standard parameterizations, the neoclassical growth model is 'near-linear' anyway so not much loss. But not always so innocuous.
- These dynamics are slightly cleaner in continuous time.

Outline

1. Intertemporal choice

2. Neoclassical growth model: planning problem
Steady state
Transitional dynamics

3. Neoclassical growth model: continuous time

Continuous Time

- Planner's problem in continuous time is to choose path of consumption per household $\boldsymbol{c} \geq 0$ to maximize

$$U(\boldsymbol{c}) = \int_0^\infty e^{-\rho t}\, u(c(t))\, dt, \qquad \rho > 0$$

subject to the flow resource constraint

$$\dot{k}(t) = f(k(t)) - \delta k(t) - c(t), \qquad k(0) > 0$$

- Current-value Hamiltonian for this problem

$$\mathcal{H}(c, k, \lambda) \equiv u(c) + \lambda(f(k) - \delta k - c)$$

- Recall that the key optimality conditions can be written

$$\mathcal{H}_c = 0, \qquad \mathcal{H}_k = \rho\lambda - \dot{\lambda}, \qquad \mathcal{H}_\lambda = \dot{k}$$

along with the transversality condition.

Hamiltonian $\mathcal{H}(c, k, \lambda) \equiv u(c) + \lambda(f(k) - \delta k - c)$

- Calculating the derivatives of the Hamiltonian

$$\mathcal{H}_c = u'(c) - \lambda$$
$$\mathcal{H}_k = \lambda(f'(k) - \delta)$$
$$\mathcal{H}_\lambda = f(k) - \delta k - c$$

- Hence our system of optimality conditions can be written

$$u'(c(t)) = \lambda(t)$$
$$\dot{\lambda}(t) = (\rho - (f'(k(t)) - \delta))\lambda(t)$$
$$\dot{k}(t) = f(k(t)) - \delta k(t) - c(t)$$

along with the transversality condition and given initial condition.

- As usual, reduce this to a system in $c(t), k(t)$ by eliminating the multipliers. To do this, differentiate the first condition with respect to t

$$u''(c(t))\dot{c}(t) = \dot{\lambda}(t)$$

Consumption Euler Equation

- Using this to eliminate the multipliers, we get the continuous time consumption Euler equation

$$\frac{\dot{c}(t)}{c(t)} = \mathcal{E}(c(t)) \times \left(f'(k(t)) - \delta - \rho\right), \qquad \mathcal{E}(c) \equiv -\frac{u'(c)}{u''(c)c}$$

  where $\mathcal{E}(c)$ is the intertemporal elasticity of substitution.

- To streamline notation, from now on we will use restrict attention to the special case of CES preferences with constant $\mathcal{E}(c) = 1/\theta$.

- With this simplification

$$\frac{\dot{c}(t)}{c(t)} = \frac{f'(k(t)) - \delta - \rho}{\theta},$$

Dynamical System

- So we have a system of two nonlinear differential equations in $c(t), k(t)$

$$\dot{c}(t) = \frac{f'(k(t)) - \delta - \rho}{\theta} c(t)$$

$$\dot{k}(t) = f(k(t)) - \delta k(t) - c(t)$$

- Two boundary conditions: (i) $k(0) > 0$ and (ii) transversality condition

$$\lim_{T \to \infty} e^{-\rho T} u'(c(T))k(T) = 0$$

- Steady state as in discrete time, $f'(k^*) = \rho + \delta$ and $c^* = f(k^*) - \delta k^*$.

- One given initial condition $k(0)$, initial consumption $c(0)$ can jump.

- Now look at dynamics local to steady state.

Local Dynamics

- Nonlinear system of the form

$$\begin{pmatrix} \dot{c}(t) \\ \dot{k}(t) \end{pmatrix} = \begin{pmatrix} \psi_1(c(t), k(t)) \\ \psi_2(c(t), k(t)) \end{pmatrix}$$

where

$$\psi_1(c, k) \equiv \frac{f'(k) - \rho - \delta}{\sigma} c, \qquad \psi_2(c, k) \equiv f(k) - \delta k - c$$

- Approximate dynamics

$$\begin{pmatrix} \dot{c}(t) \\ \dot{k}(t) \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial c}\psi_1(c, k) & \frac{\partial}{\partial k}\psi_1(c, k) \\ \frac{\partial}{\partial c}\psi_2(c, k) & \frac{\partial}{\partial k}\psi_2(c, k) \end{pmatrix} \begin{pmatrix} c(t) - c^* \\ k(t) - k^* \end{pmatrix}$$

where the Jacobian matrix is evaluated at steady state c^*, k^*.

- Local stability depends on signs of the *eigenvalues* of this Jacobian.

Local Dynamics

- Calculating the elements of the Jacobian

$$\boldsymbol{J} = \left(\begin{array}{cc} \frac{\partial}{\partial c}\psi_1(c,k) & \frac{\partial}{\partial k}\psi_1(c,k) \\ \\ \frac{\partial}{\partial c}\psi_2(c,k) & \frac{\partial}{\partial k}\psi_2(c,k) \end{array} \right)\Bigg|_{c^*,k^*} = \left(\begin{array}{cc} 0 & \dfrac{f''(k^*)}{\theta}c^* \\ \\ -1 & \rho \end{array} \right)$$

- In slight abuse of notation, let λ_1, λ_2 denote the eigenvalues of this matrix.

- Eigenvalues characterized by determinant

$$\det(\boldsymbol{J}) = \lambda_1\lambda_2 = \frac{f''(k^*)}{\theta}c^* < 0$$

and trace

$$\mathrm{tr}(\boldsymbol{J}) = \lambda_1 + \lambda_2 = \rho > 0$$

- Hence eigenvalues real and of opposite sign, say

$$\lambda_1 < 0 < \lambda_2$$

and so, as anticipated, imply saddle path dynamics.

Eigenvalues

- Eigenvalues are roots of the characteristic polynomial, here a quadratic

$$0 = \lambda^2 - \mathrm{tr}(\boldsymbol{J})\lambda + \det(\boldsymbol{J}) = \lambda^2 - \rho\lambda + \frac{f''(k^*)}{\theta}c^*$$

- Gives

$$\lambda_1 = \frac{\rho - \sqrt{\rho^2 - 4\frac{f''(k^*)c^*}{\theta}}}{2} < 0 < \frac{\rho + \sqrt{\rho^2 - 4\frac{f''(k^*)c^*}{\theta}}}{2} = \lambda_2$$

- Both of these roots have an economic interpretation.

- Stable root $\lambda_1 < 0$ controls *speed of adjustment* to steady state c^*, k^*.

- Unstable root $\lambda_2 > 0$ gives the *slope of the stable* arm local to c^*, k^*, that is, the slope of the policy function at steady state.

Method of Undetermined Coefficients

- Direct approach to solving the linearized dynamics.

- Write out the linear approximation

$$\dot{c}(t) = \frac{f''(k^*)c^*}{\theta}(k(t) - k^*)$$

and

$$\dot{k}(t) = \rho(k(t) - k^*) - (c(t) - c^*)$$

- Implies a *second-order differential* equation in $k(t)$, namely

$$\ddot{k}(t) = \rho\dot{k}(t) - \frac{f''(k^*)c^*}{\theta}(k(t) - k^*)$$

- Now *guess* model is solved by a linear law of motion

$$\dot{k}(t) = \lambda(k(t) - k^*)$$

for some coefficient λ to be determined.

Method of Undetermined Coefficients

- Guess implies that

$$\ddot{k}(t) = \lambda \dot{k}(t) = \lambda^2 (k(t) - k^*)$$

- So if this guess is to be true, λ must satisfy

$$\left[\lambda^2 - \rho\lambda + \frac{f''(k^*)c^*}{\theta} \right] (k(t) - k^*) = 0$$

- Has to hold for *any* value of $k(t) - k^*$, which is only true if

$$\lambda^2 - \rho\lambda + \frac{f''(k^*)c^*}{\theta} = 0$$

- But this is just the characteristic polynomial again.

- So choose the stable root $\lambda \equiv \lambda_1 < 0$, approximate solution

$$\dot{k}(t) = \lambda_1 (k(t) - k^*) \qquad \Rightarrow \qquad k(t) = k^* + e^{\lambda_1 t}(k(0) - k^*)$$

Stable Arm

- Then recover consumption from linearized resource constaint

$$\dot{k}(t) = \lambda_1(k(t) - k^*) \qquad \Rightarrow \qquad c(t) - c^* = (\rho - \lambda_1)(k(t) - k^*)$$

- Stable arm is a curve $c = g(k)$ with $c^* = g(k^*)$, so local to steady state

$$c(t) - c^* = g'(k^*)(k(t) - k^*)$$

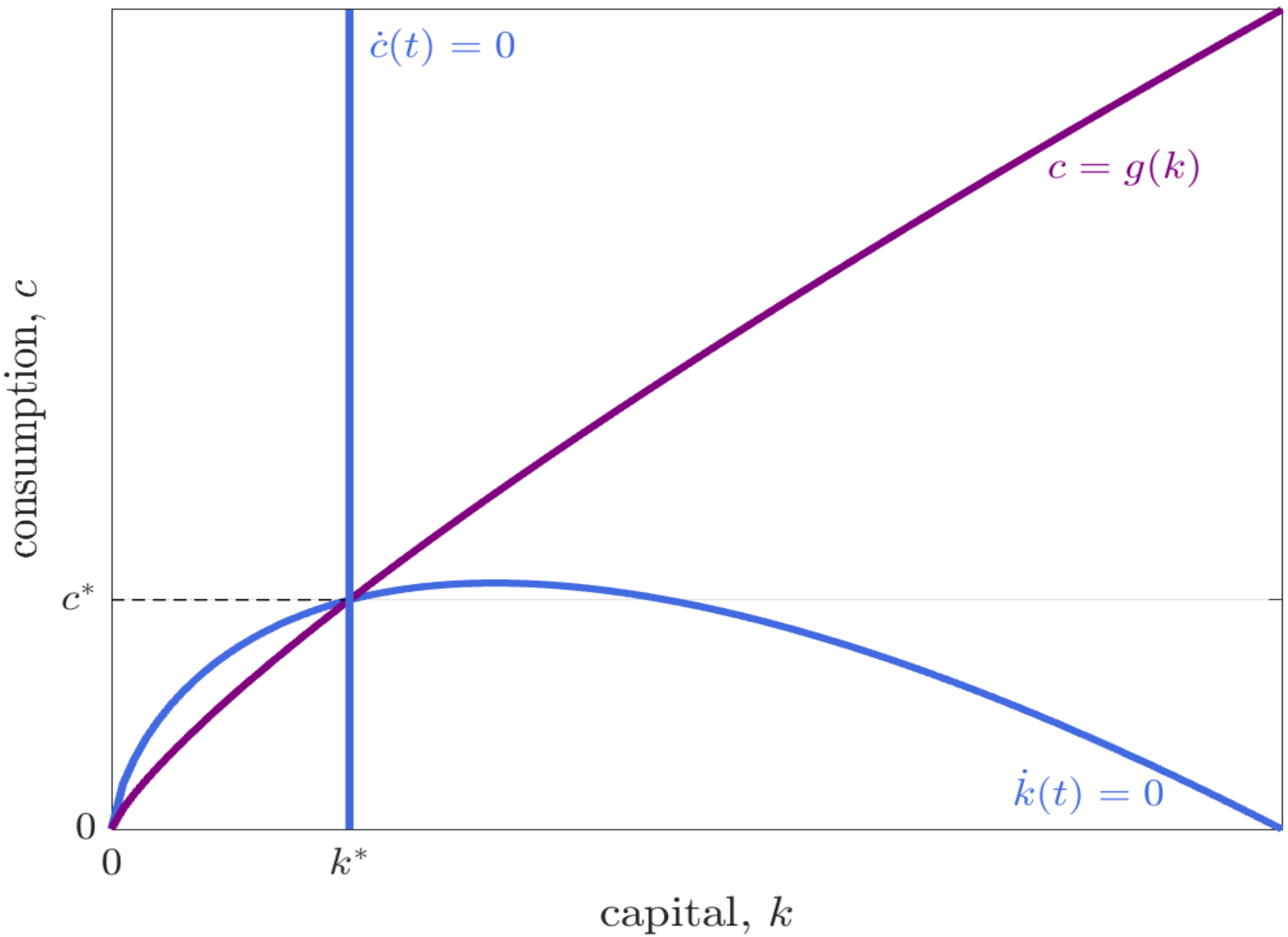
- So matching coefficients, the slope of the stable arm local to c^*, k^* is

$$g'(k^*) = \rho - \lambda_1 > \rho > 0$$

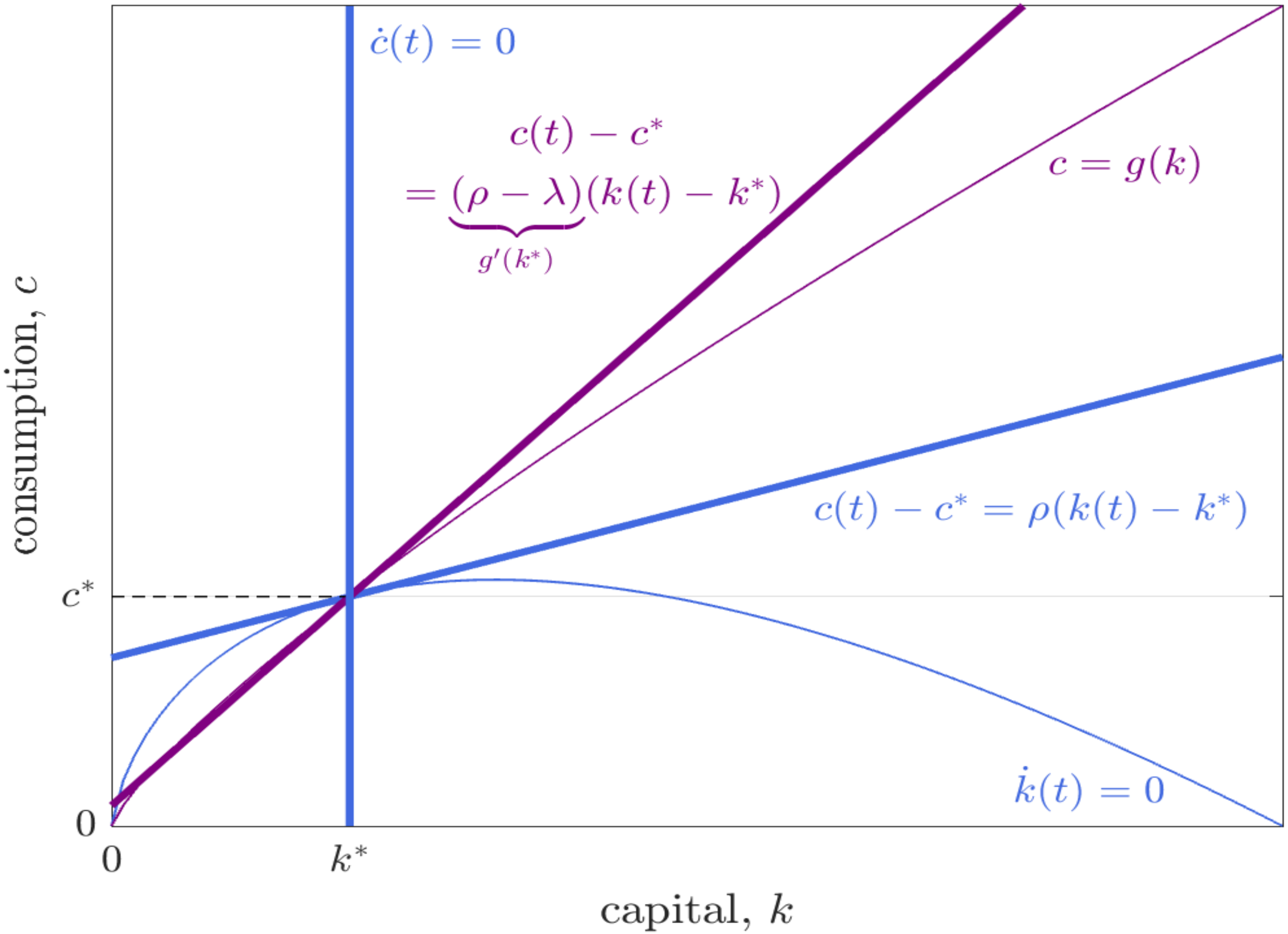
- But we calculated $\mathrm{tr}(\boldsymbol{J}) = \rho$ and since $\mathrm{tr}(\boldsymbol{J}) = \lambda_1 + \lambda_2$ we conclude

$$g'(k^*) = \lambda_2 > 0$$

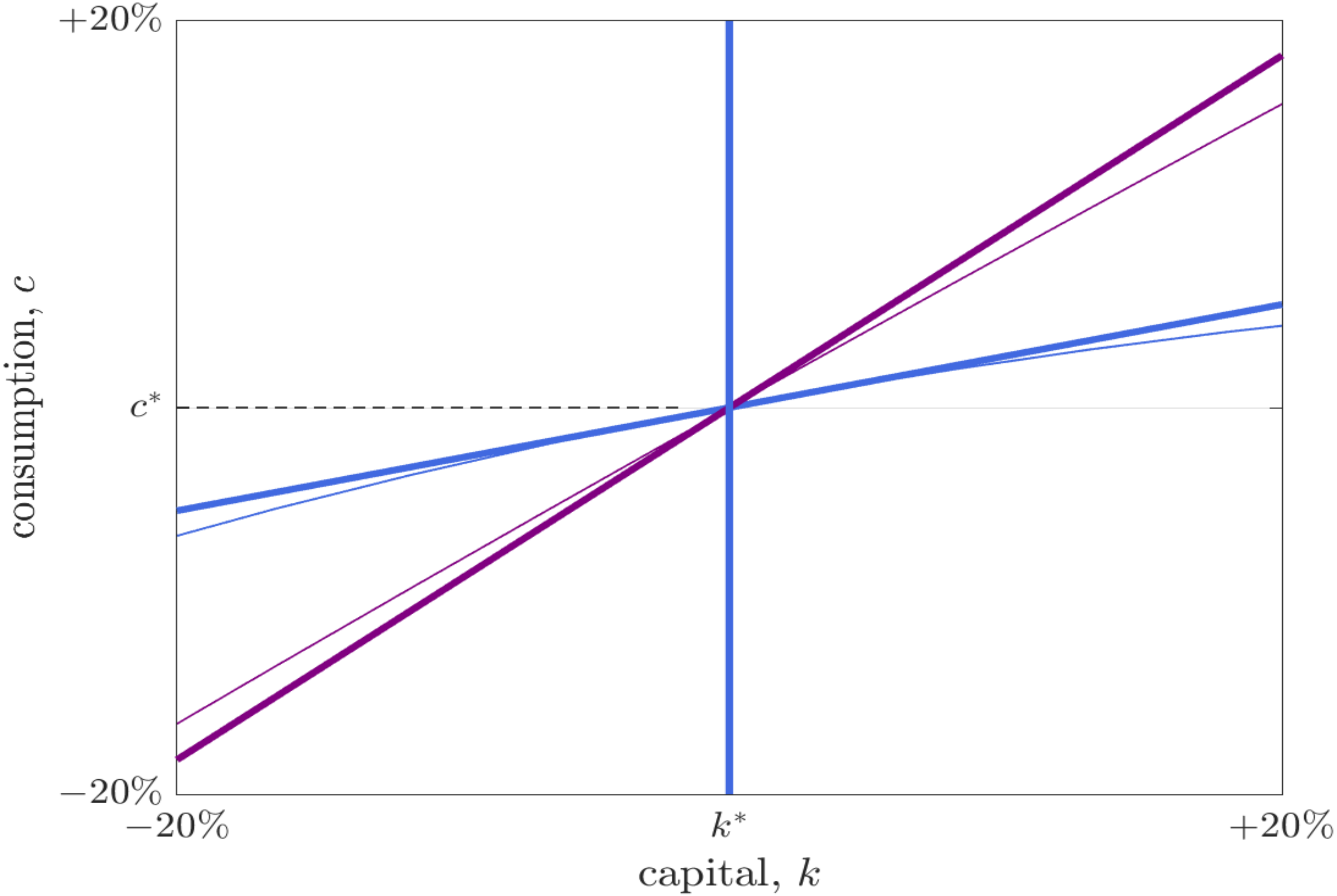
True Dynamics



Local Approximation to True Dynamics



Zooming In



Taking Stock

- Have set up and determined basic properties of neoclassical growth model.

- Now need to actually put this to work.

Next class

- Restriction on preferences needed to ensure balanced growth.

- Applications, examples, extensions.

Homework

- Consider the discrete time neoclassical growth model with $u(c) = \log c$, full depreciation $\delta = 1$, and $y = f(k) = k^\alpha A^{1-\alpha}$.

- CHECK. Show that the consumption policy function

$$c_t = g(k_t) = (1 - \alpha\beta)f(k_t)$$

  satisfies the consumption Euler equation and resource constraint on every date t, for any given $k_0 > 0$, and satisfies the transversality condition.

- Consider the continuous time neoclassical growth model. Local to the steady state c^*, k^* the policy function $c = g(k)$ is approximately

$$c(t) - c^* = g'(k^*)(k(t) - k^*), \qquad g'(k^*) = \lambda > 0$$

- CHECK. Use the method of undetermined coefficients to show that $\lambda > 0$ is the unstable root of the local dynamics.