Economic Growth

Lecture 3: Solow model, part two

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Outline

1. **Solow model in continuous time**

2. **A first look at sustained growth: the AK model**

3. **Balanced growth and different types of technological change**

4. **Uzawa's theorem and labor-augmenting technological change**

5. **Solow model with technological change**

Towards Continuous Time

- Period length $\Delta > 0$ in units of calendar time
- Periods $t = 0, \Delta, 2\Delta, 3\Delta, \ldots$
- All *flows* multiplied by period length, so for example

$$K_{t+\Delta} - K_t = I_t\Delta - \delta\Delta K_t$$

- Let exogenous productivity A_t and labor L_t grow according to

$$A_{t+\Delta} = e^{g\Delta}A_t$$

$$L_{t+\Delta} = e^{n\Delta}L_t$$

(in anticipation of continuously-compounded growth rates)

Towards Continuous Time

- Divide by $\Delta > 0$ to get

$$\frac{K_{t+\Delta} - K_t}{\Delta} = I_t - \delta K_t$$

  with

$$\frac{A_{t+\Delta} - A_t}{\Delta} = \frac{e^{g\Delta} - 1}{\Delta} A_t$$
$$\frac{L_{t+\Delta} - L_t}{\Delta} = \frac{e^{n\Delta} - 1}{\Delta} L_t$$

- Take limit as period length shrinks $\Delta \to 0$. From l'Hôpital's rule,

$$\lim_{\Delta \to 0} \frac{e^{x\Delta} - 1}{\Delta} = x$$

  (or can use $e^{x\Delta} \approx 1 + x\Delta$)

Continuous Time Limit

- So as period length $\Delta \to 0$ we get

$$\dot{K}(t) \equiv \frac{dK(t)}{dt} = I(t) - \delta K(t)$$

and

$$\dot{A}(t) \equiv \frac{dA(t)}{dt} = gA(t), \qquad \dot{L}(t) \equiv \frac{dL(t)}{dt} = nL(t)$$

- Productivity $A(t)$ and labor $L(t)$ grow exponentially at rates g and n

$$\frac{\dot{A}(t)}{A(t)} = g \quad \Leftrightarrow \quad \frac{d}{dt}\log A(t) = g \quad \Rightarrow \quad A(t) = e^{gt}A(0)$$

$$\frac{\dot{L}(t)}{L(t)} = n \quad \Leftrightarrow \quad \frac{d}{dt}\log L(t) = n \quad \Rightarrow \quad L(t) = e^{nt}L(0)$$

Solow Model in Continuous Time

- Time $t \geq 0$
- Capital accumulation
$$\dot{K}(t) = I(t) - \delta K(t)$$
- Exogenous productivity and labor force
$$\dot{A}(t) = gA(t), \qquad \dot{L}(t) = nL(t)$$
- Constant savings rate
$$I(t) = S(t) = sY(t) = sF(K(t), L(t), A(t))$$
- Aggregate production function $F(K, L, A)$ satisfies Assumptions 1 and 2.
- Law of motion in continuous time
$$\dot{K}(t) = sF(K(t), L(t), A(t)) - \delta K(t)$$
A *nonlinear differential equation* in $K(t)$ given exogenous $A(t), L(t)$.

Special Case: Constant A

- To fix ideas, consider special case with $A(t) = 1$ [i.e., $g = 0, A(0) = 1$]

- **Long-run productivity growth needs special attention.**

- Then write output per worker, capital per worker

$$y(t) \equiv \frac{Y(t)}{L(t)}, \qquad k(t) \equiv \frac{K(t)}{L(t)}$$

- So growth in capital per worker is

$$\frac{\dot{k}(t)}{k(t)} = \frac{\dot{K}(t)}{K(t)} - \frac{\dot{L}(t)}{L(t)} = \frac{\dot{K}(t)}{K(t)} - n$$

Special Case: Constant A

- Using law of motion for capital and constant returns to scale in K, L

$$\frac{\dot{k}(t)}{k(t)} = \frac{\dot{K}(t)}{K(t)} - n = s\frac{F(K(t), L(t), 1)}{K(t)} - (\delta + n)$$

$$= s\frac{f(k(t))}{k(t)} - (\delta + n)$$

- Hence law of motion in intensive form is

$$\dot{k}(t) = sf(k(t)) - (\delta + n)k(t)$$

- Same as before but with effective depreciation rate $\delta + n$.

Steady State

- Steady states k^* where $\dot{k}(t) = 0$.

- Unique non-trivial steady state $k^* > 0$ where capital/output ratio is

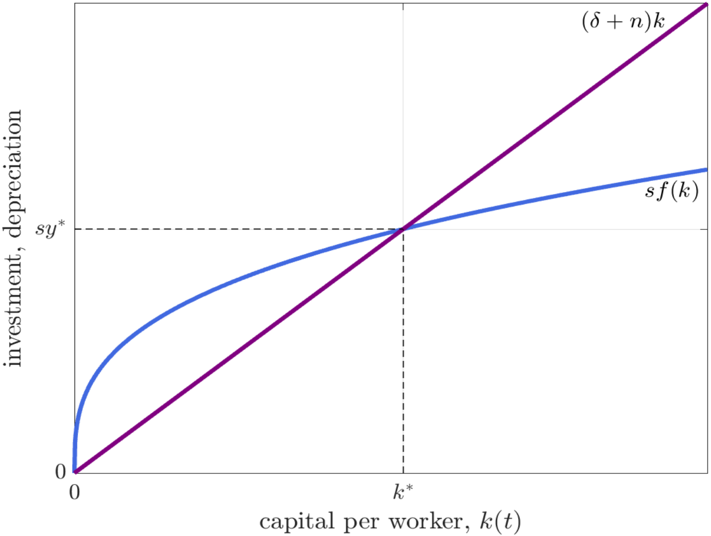
$$\frac{k^*}{y^*} = \frac{k^*}{f(k^*)} = \frac{s}{\delta + n}$$

- Pins down steady state output and consumption per worker

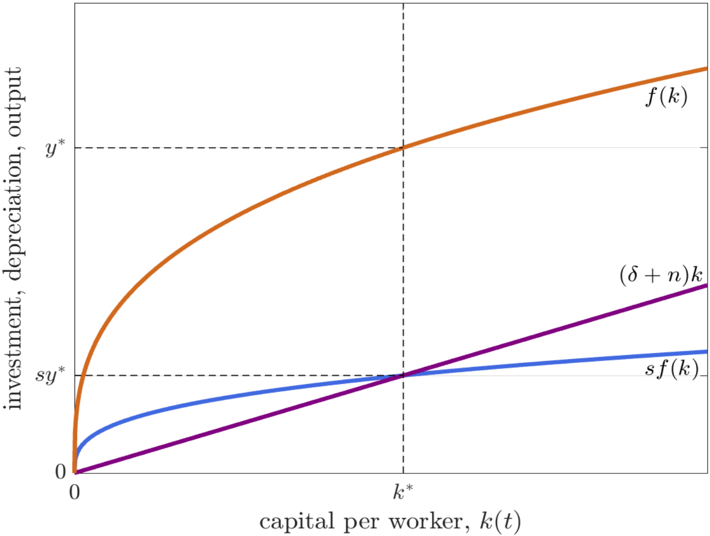
$$y^* = f(k^*), \qquad c^* = (1-s)f(k^*)$$

- Comparative statics same as in discrete time model.

Solow Diagram



Solow Diagram



Transitional Dynamics: Global Stability

- Write growth rate of capital/labor ratio

$$\gamma(k(t)) \equiv \frac{\dot{k}(t)}{k(t)} = s\frac{f(k(t))}{k(t)} - (\delta + n) = \frac{s}{h(k(t))} - (\delta + n)$$

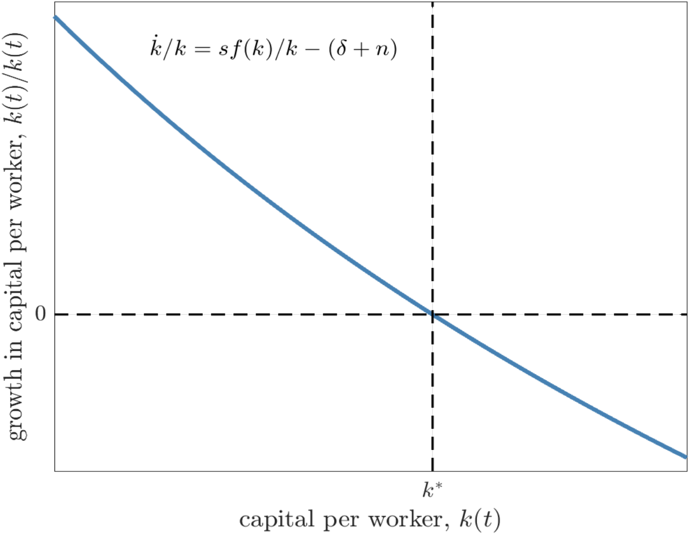
  where $h(k) = k/f(k)$ again denotes the capital/output ratio.

- Since $h(k)$ is strictly increasing in k and satisfies $h(k^*) = s/(\delta + n)$, the growth rate $\gamma(k)$ is strictly decreasing in k with $\gamma(k) > 0$ iff $k < k^*$.

- So for $k(t) < k^*$ the capital stock grows, $\gamma(k(t)) > 0$, but at a diminishing rate, with $\gamma(k(t)) \searrow 0$ as $k(t) \nearrow k^*$.

- And for $k(t) > k^*$ the capital stock shrinks, $\gamma(k(t)) < 0$, but at a diminishing rate, with $\gamma(k(t)) \searrow 0$ as $k(t) \searrow k^*$.

Transitional Dynamics: Phase Diagram



Transitional Dynamics: Speed of Convergence

- Write differential equation

$$\dot{k}(t) = sf(k(t)) - (\delta + n)k(t) \equiv \psi(k(t))$$

- Consider a *linear approximation* to $\psi(k)$ around k^*

$$\dot{k}(t) \approx \psi(k^*) + \psi'(k^*)(k(t) - k^*) = \psi'(k^*)(k(t) - k^*)$$

- Treat as exact and solve linear differential equation to get

$$k(t) = k^* + e^{-\lambda t}(k(0) - k^*), \qquad \lambda \equiv -\psi'(k^*) > 0$$

- A weighted average of initial $k(0)$ and steady state k^* with weights $e^{-\lambda t}$ on $k(0)$ decaying at rate λ as $t \to \infty$.

- Speed of convergence $\lambda \equiv -\psi'(k^*)$ pinned down by parameters $s, \delta, f'(k)$. Rapid convergence if λ is large, slow convergence if λ close to zero.

Transitional Dynamics: Speed of Convergence

- Speed of convergence λ given by

$$\lambda \equiv -\psi'(k^*) = -\frac{sf'(k^*)k^*}{k^*} + (\delta + n)$$

$$= -\frac{f'(k^*)k^*}{f(k^*)}(\delta + n) + (\delta + n)$$

$$= +\left(1 - \frac{f'(k^*)k^*}{f(k^*)}\right)(\delta + n) > 0$$

- Speed of convergence determined by (i) the *degree of concavity* in the production function and (ii) the effective depreciation rate.

- Example. Cobb-Douglas production function $f(k) = k^\alpha$ for $\alpha \in (0, 1)$.

$$\lambda = (1 - \alpha)(\delta + n) > 0$$

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The AK Model

- Slow convergence when production function is near linear.

- Limiting case with output linear in capital is known as the AK model

$$Y(t) = AK(t), \qquad A > 0$$

- Remark. Does not satisfy Assumptions 1 and 2. No diminishing returns to capital. Sustained growth is a possibility but is not inevitable.

- Similar with $Y(t) = AK(t) + BL(t)$ but we will stick with AK version.

- Many growth models are 'asymptotically' AK.

The AK Model

- Law of motion for capital per worker is then

$$\dot{k}(t) = sAk(t) - (\delta + n)k(t) = (sA - (\delta + n))k(t)$$

- So the growth rate of capital per worker is constant

$$\frac{\dot{k}(t)}{k(t)} = sA - (\delta + n)$$

- Remark. No non-trivial steady state $k^* > 0$ except in 'knife-edge' case $sA = (\delta + n)$ exactly, in which case every $k > 0$ is a steady state.

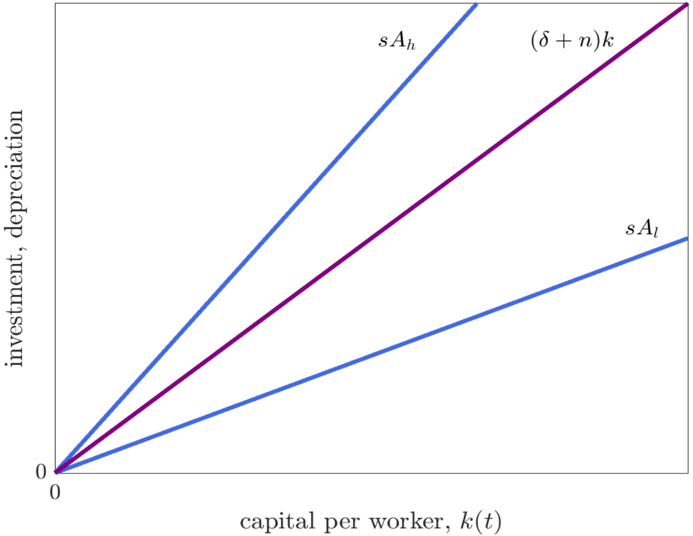
- Solving the linear differential equation

$$k(t) = \exp((sA - (\delta + n))\, t)\; k(0)$$

hence

$$y(t) = A \exp((sA - (\delta + n))\, t)\; k(0)$$

Solow Diagram: AK Model



Sustained Growth?

- There is sustained growth if $sA > (\delta + n)$, economy grows without bound.

- But if $sA < (\delta + n)$ economy shrinks asymptotically to $k = 0$.

- A 'two-edged' sword. No *ceiling* on capital accumulation if A is sufficiently high. But no *floor* under capital decumulation if A is sufficiently low.

- For usual Solow model, Assumptions 1 and 2 imply long-run levels independent of initial condition $k(0)$.

- But in AK model, long-run levels depend *permanently* on initial $k(0)$.

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Kaldor Facts

- Production function $F(K, L, A)$ is (still) *too general*.

- Kaldor (1963) observed that long run growth features
  - trend in output per person and real wages
  - but no trend in capital/output ratio, real interest rates, or factor shares

- As a simple starting point, we will restrict production function further to be consistent with these '*stylized facts*'.

- An economy that grows in a manner consistent with these facts is often said to exhibit *balanced growth*.

Balanced Growth

- Will focus on two key features of balanced growth:

  (i) asymptotic constancy of the factor income shares

$$\alpha_K(t) \equiv \frac{R(t)K(t)}{Y(t)}, \qquad \text{and} \qquad \alpha_L(t) \equiv \frac{w(t)L(t)}{Y(t)}$$

  (ii) asymptotic constancy of the capital/output ratio $K(t)/Y(t)$

- REMARK. Since the production function has constant returns to scale in K, L and factors are (implicitly) priced competitively, $\alpha_K(t) + \alpha_L(t) = 1$.

- REMARK. If the factor shares and capital/output ratio are asymptotically constant, then (a) so is the rental rate $R(t)$, and (b) the real wage $w(t)$ grows at the same rate as output per worker $Y(t)/L(t)$.

Three Types of Neutral Technological Change

- Let $\tilde{F}(K, L, A)$ be some production function with CRS in K, L.

- Technological change is *Hicks-neutral* if

$$\tilde{F}(K, L, A) = AF(K, L)$$

Hicks-neutral A has no effect on marginal rates of technical substitution, $\tilde{F}_K/\tilde{F}_L = F_K/F_L$ independent of A. Shifts in isoquants, but no effect on curvature.

- Technological change is *Solow-neutral* (capital-augmenting) if

$$\tilde{F}(K, L, A) = F(AK, L)$$

'As if' more capital. Y/L independent of A at any given $w = \tilde{F}_L$.

- Technological change is *Harrod-neutral* (labor-augmenting) if

$$\tilde{F}(K, L, A) = F(K, AL)$$

'As if' more labor. K/Y independent of A at any given $R = \tilde{F}_K$.

Harrod-Neutral Change *is* Labor-Augmenting

- Suppose capital per worker $k = K/L$ is given by the marginal condition

$$\tilde{F}_K(K, L, A) = R \qquad \Rightarrow \qquad k(R, A)$$

- Implies output per worker and capital output ratio

$$y(R, A) = \tilde{F}(k(R, A), 1, A), \qquad \text{and} \qquad h(R, A) = \frac{k(R, A)}{y(R, A)}$$

- Technological change is *Harrod-neutral* if $h(R, A)$ is independent of A.

- ROBINSON'S THEOREM. The production function $\tilde{F}(K, L, A)$ is Harrod-neutral if and only if it can be written

$$\tilde{F}(K, L, A) = F(K, AL)$$

- Similarly for Solow-neutral change.

Three Types of Neutral Technological Change

- Can of course have vector $\boldsymbol{A} = (A_H, A_K, A_L)$ and production function

$$\tilde{F}(K, L, \boldsymbol{A}) = A_H F(A_K K, A_L L)$$

- Still restrictive since technological change could transform F entirely.

- But we have more important matters to consider.

- Uzawa (1961) showed that balanced growth requires that technological change takes one specific form — *must be labor-augmenting.*

- To establish this result, write production function

$$\tilde{F}(K, L;\ t)$$

  for some type of technological change to be determined.

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Uzawa's Theorem

- This version follows Schlicht (2006) and Jones and Scrimgeour (2008).

- Setup. Suppose $\tilde{F}(K, L; t)$ has CRS in K, L, and that

$$Y(t) = C(t) + I(t), \qquad \dot{K}(t) = I(t) - \delta K(t), \qquad \dot{L}(t) = nL(t)$$

  Now suppose there exists some $\tau \geq 0$ such that for all $t \geq \tau$

$$\frac{\dot{K}(t)}{K(t)} = g_K > 0, \qquad \frac{\dot{Y}(t)}{Y(t)} = g_Y > 0, \qquad \frac{\dot{C}(t)}{C(t)} = g_C > 0$$

- Uzawa's Theorem. If $C(t), I(t) > 0$ for all $t \geq \tau$, then

  (i) $g_K = g_Y = g_C$, and

  (ii) the production function $\tilde{F}$ can be represented by

$$\tilde{F}(K(t), L(t); t) = F(K(t), A(t)L(t)), \qquad t \geq \tau$$

  where

$$\frac{\dot{A}(t)}{A(t)} = g = g_Y - n$$

Uzawa's Theorem: Part (i)

- Proof (Sketch). For (i), use the capital accumulation equation to write

$$g_K = \frac{I(t)}{K(t)} - \delta, \qquad t \geq \tau$$

So along such a path we must have

$$g_I = g_K$$

Differentiating the resource constraint with respect to t once

$$g_Y = \frac{C(t)}{Y(t)} g_C + \frac{I(t)}{Y(t)} g_I$$

and then differentiating with respect to t again

$$0 = (g_C - g_Y) g_C \frac{C(t)}{Y(t)} + (g_I - g_Y) g_I \frac{I(t)}{Y(t)}$$

Since $C(t), I(t) > 0$ we conclude $g_C = g_Y$ and $g_Y = g_I = g_K$ for $t \geq \tau$.

Uzawa's Theorem: Part (ii)

- For (ii), write output at date $t = \tau$

$$Y(\tau) = \tilde{F}(K(\tau), L(\tau); \tau)$$

Use $Y(t) = e^{g_Y(t-\tau)}Y(\tau)$ etc to write

$$Y(t)e^{-g_Y(t-\tau)} = \tilde{F}(K(t)e^{-g_K(t-\tau)}, L(t)e^{-n(t-\tau)}; \tau), \qquad t \geq \tau$$

Since $\tilde{F}$ has CRS in K, L, divide through by exponential factor to get

$$Y(t) = \tilde{F}(K(t)e^{(g_Y-g_K)(t-\tau)}, L(t)e^{(g_Y-n)(t-\tau)}; \tau), \qquad t \geq \tau$$

But for $t \geq \tau$ from part (i) we know $g_Y = g_K$ so this can be written

$$Y(t) = F(K(t), A(t)L(t)), \qquad t \geq \tau$$

where $A(t) = e^{g(t-\tau)}$ and $g = g_Y - n$.

- In short, technological change has a labor-augmenting representation.

Intuition

- The underlying asymmetry here is the capital is accumulated out of output while labor is not.

- Use CRS to write the production function

$$1 = \tilde{F}\left(\frac{K(t)}{Y(t)}, \frac{L(t)}{Y(t)}\,;\, t\right)$$

- Along a balanced growth path, the capital/output ratio is constant, capital $K(t)$ inherits its trend from $Y(t)$.

- But labor $L(t)$ is exogenous, does not inherit a trend from $Y(t)$.

- So technology has to be 'as if' labor is increasing, enters in labor-augmenting form $A(t)L(t)$, compensates for underlying asymmetry.

Implications for Factor Payments

- With the labor-augmenting representation, capital's share is

$$\alpha_K = \frac{RK}{Y} = F_K(K, AL)\frac{K}{Y}$$

- Since F is homogeneous of degree 1 in K, L (has CRS), by Euler's theorem on homogeneous functions the derivative F_K is homogenous of degree zero

$$F_K(K, AL) = F_K\left(\frac{K}{AL}, 1\right)$$

- So along a balanced growth path where $g_Y = g_K = g + n$ both the capital/output ratio and the marginal product of capital are constant, hence capital's share α_K and labor's share $\alpha_L = 1 - \alpha_K$ are also constant.

- So along a balanced growth path the marginal product of labor grows at the same rate as output per worker, $g = g_Y - n$.

Discussion

- Remark. Does not require $\tilde{F}(K, L; t) = F(K, AL)$, instead requires that that $\tilde{F}$ can be *representated* this way.

- Consider a Cobb-Douglas production function

$$F(K, L; \boldsymbol{A}) = A_H (A_K K)^\alpha (A_L L)^{1-\alpha}$$

All three types of technological change but has the representation

$$F(K, L; \boldsymbol{A}) = K^\alpha (AL)^{1-\alpha},$$

where

$$A \equiv \left(A_H A_K^\alpha A_L^{1-\alpha} \right)^{\frac{1}{1-\alpha}}$$

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Solow Model with Technological Change

- Time $t \geq 0$

- Capital accumulation
$$\dot{K}(t) = I(t) - \delta K(t)$$

- Exogenous productivity and labor force
$$\dot{A}(t) = gA(t), \qquad \dot{L}(t) = nL(t)$$

- In anticipation of balanced growth, labor-augmenting productivity
$$Y(t) = F(K(t), A(t)L(t))$$

- Constant savings rate
$$I(t) = S(t) = sY(t)$$

- Law of motion in continuous time
$$\dot{K}(t) = sF(K(t), A(t)L(t)) - \delta K(t)$$

Solow Model with Technological Change

- Write output *per effective worker*, capital *per effective worker*

$$y(t) \equiv \frac{Y(t)}{A(t)L(t)}, \qquad k(t) \equiv \frac{K(t)}{A(t)L(t)}$$

- So growth in capital per effective worker is

$$\begin{aligned}\frac{\dot{k}(t)}{k(t)} &= \frac{\dot{K}(t)}{K(t)} - (g+n) \\[2em] &= s\frac{F(K(t), A(t)L(t))}{K(t)} - (\delta + g + n) \\[2em] &= s\frac{f(k(t))}{k(t)} - (\delta + g + n)\end{aligned}$$

Solow Model with Technological Change

- Hence law of motion in intensive form is

$$\dot{k}(t) = sf(k(t)) - (\delta + g + n)k(t)$$

- Same as before but with effective depreciation rate $\delta + g + n$.

- Unique non-trivial steady state k^* where capital/output ratio is

$$\frac{k^*}{y^*} = \frac{k^*}{f(k^*)} = \frac{s}{\delta + g + n}$$

- Pins down steady state output and consumption per effective worker

$$y^* = f(k^*), \qquad c^* = (1 - s)f(k^*)$$

- Steady state k^* globally stable, $k(t) \to k^*$, following usual arguments.

Balanced Growth Path

- Convergence $k(t) \to k^*$ constant implies long-run capital per worker

$$\frac{K(t)}{L(t)} \to k^* A(t)$$

- Convergence $y(t) \to y^*$ constant implies long-run output per worker

$$\frac{Y(t)}{L(t)} \to y^* A(t)$$

- Capital per worker and output per worker both proportional to $A(t)$ so capital/output ratio constant.

- Capital per worker and output per worker both grow at rate g.

- From now on, use 'steady state' and 'balanced growth' interchangeably. Only a question of whether using detrended variables or not.

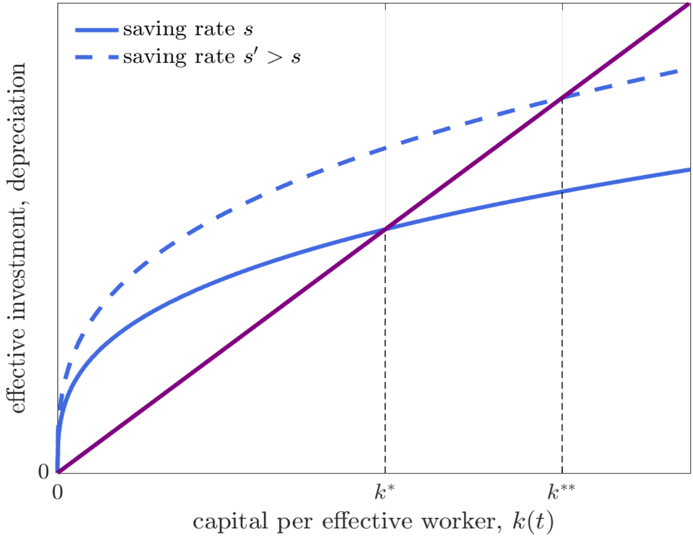
Policy Experiment: Increase in Saving

- Consider economy in steady state k^* with saving rate s.
- Suppose at $t = 0$ a permanent increase in saving rate from s to $s' > s$.
  - capital per effective worker stock is unchanged at $k(0) = k^*$, capital is 'predetermined' (a state variable)
  - hence output per effective worker is unchanged at $y(0) = f(k^*)$
  - investment per effective worker immediately increases to $s'f(k^*) > sf(k^*)$ while consumption per effective worker falls
  - so capital begins to accumulate

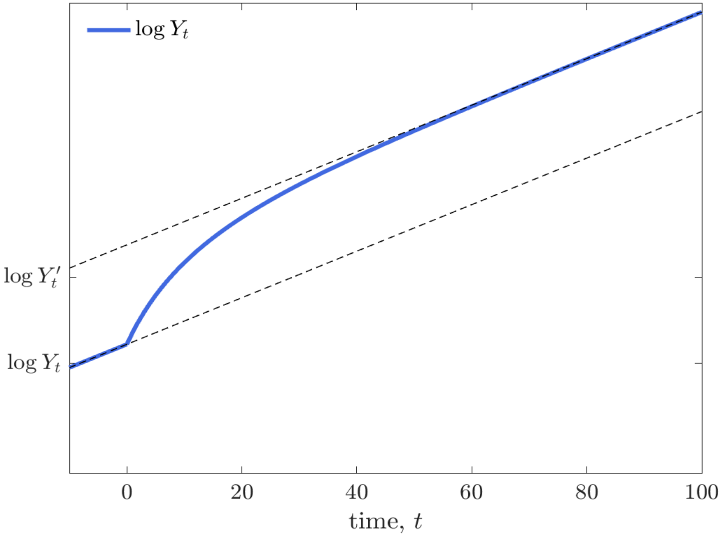
$$\dot{k}(0) = s'f(k^*) - (\delta + g + n)k^* = (s' - s)f(k^*) > 0$$

- In short run, economy grows faster than underlying trend as capital accumulates faster than normal.
- But in long run, $k(t) \to k^{**} > k^*$ and growth slows back to normal.

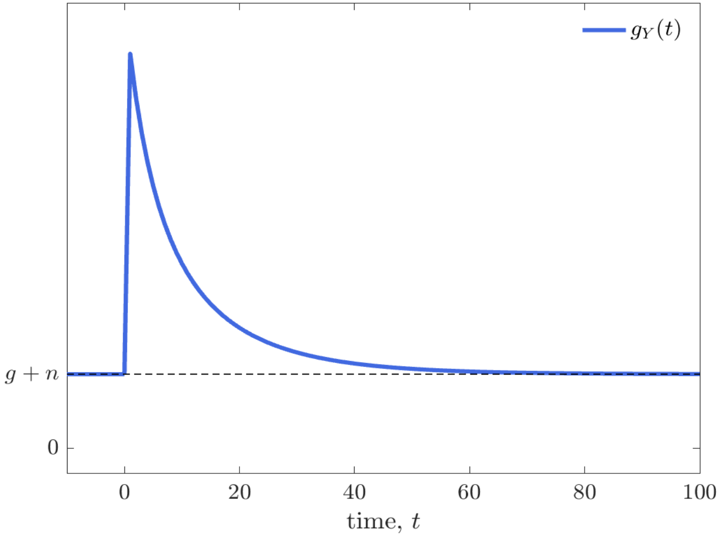
Policy Experiment: Increase in Saving



Long Run Level Effect



But No Long Run Growth Effect



Summary

- Simple, tractable framework. Clarifies meaning of *short run*, *long run* etc.

- Emphasizes distinction between *level effects* and *growth effects*.

  – increase in s has long-run level effect on Y/L but no long run growth effect

- All long run growth in output per worker is exogenous, due to A.

- Savings s is also exogenous.

- Most of the rest of this course: trying to crack open these 'black boxes'.

- Bur first: some empirics, putting the Solow model to work.

Next class

- Using the Solow model to interpret cross-country data.

- Growth accounting and development accounting.

Homework

- Suppose the production function is Cobb-Douglas

$$y = f(k) = k^{\alpha}, \qquad 0 < \alpha < 1$$

and that the law of motion for capital per effective worker is

$$\dot{k}(t) = sf(k(t)) - (\delta + g + n)k(t), \qquad k(0) > 0 \text{ given}$$

- Check. Show that the *exact* solution for the time-path of $k(t)$ is

$$k(t) = \left(e^{-\lambda t}k(0)^{1-\alpha} + (1 - e^{-\lambda t})k^{*1-\alpha}\right)^{\frac{1}{1-\alpha}}, \qquad t \geq 0$$

where $\lambda \equiv (1-\alpha)(\delta + g + n) > 0$ is the speed of convergence.

Hint: show that the law of motion for capital per effective worker implies a *linear* differential equation in the capital/output ratio. Solve it.