Economic Growth

Lecture 2: Solow model, part one

Chris Edmond

MIT 14.452

Fall 2021

Solow Model: Introduction

- Benchmark model of economic growth, capital accumulation
- Earlier Harrod-Domar growth models
  - capital and labor used in *fixed proportions* ('Leontieff technologies')
  - excess labor may be unemployed
  - balanced growth is a 'knife-edge' property
- Solow (1956) model
  - 'neoclassical' aggregate production function with *smooth substitution* between capital and labor
  - full employment
  - balanced growth is a robust property

Outline

1. **Setup**
   - Aggregate production function
   - Constant saving rate
2. **Special case with constant A and L**
   - Steady state
   - Transitional dynamics
   - Comparative statics and the 'golden rule'
3. **Decentralization**

Setup

- Discrete time $t = 0, 1, 2, \ldots$

- Single final good, can be consumed or transformed into capital

- Output of the final good is given by an *aggregate production function*

$$Y_t = F(K_t, L_t, A_t)$$

  where K_t denotes physical capital, L_t denotes labor and A_t parameterizes the level of productivity, 'technology' broadly speaking.

- Closed economy, no government sector

$$Y_t = C_t + I_t$$

- Capital depreciates at constant rate δ per period

$$K_{t+1} - K_t = I_t - \delta K_t, \qquad 0 < \delta < 1, \qquad K_0 > 0$$

Outline

1. Setup
  Aggregate production function
  Constant saving rate

2. Special case with constant A and L
  Steady state
  Transitional dynamics
  Comparative statics and the 'golden rule'

3. Decentralization

Aggregate Production Function

- **Assumption 1.** The aggregate production function $F(K, L, A)$ is twice continuously differentiable in K and L with:

  (i) *positive marginal products*

$$F_K(K, L, A) > 0, \qquad \text{and} \qquad F_L(K, L, A) > 0$$

  (ii) *diminishing returns* to K and L

$$F_{KK}(K, L, A) < 0, \qquad \text{and} \qquad F_{LL}(K, L, A) < 0$$

  (iii) *constant returns to scale* in K and L

$$F(xK, xL, A) = xF(K, L, A) \qquad \text{for any } x > 0$$

- **Remark.** Then by Euler's theorem on homogeneous functions, the aggregate production function can also be written

$$F(K, L, A) = F_K(K, L, A)K + F_L(K, L, A)L$$

Aggregate Production Function

- We will often make use of:

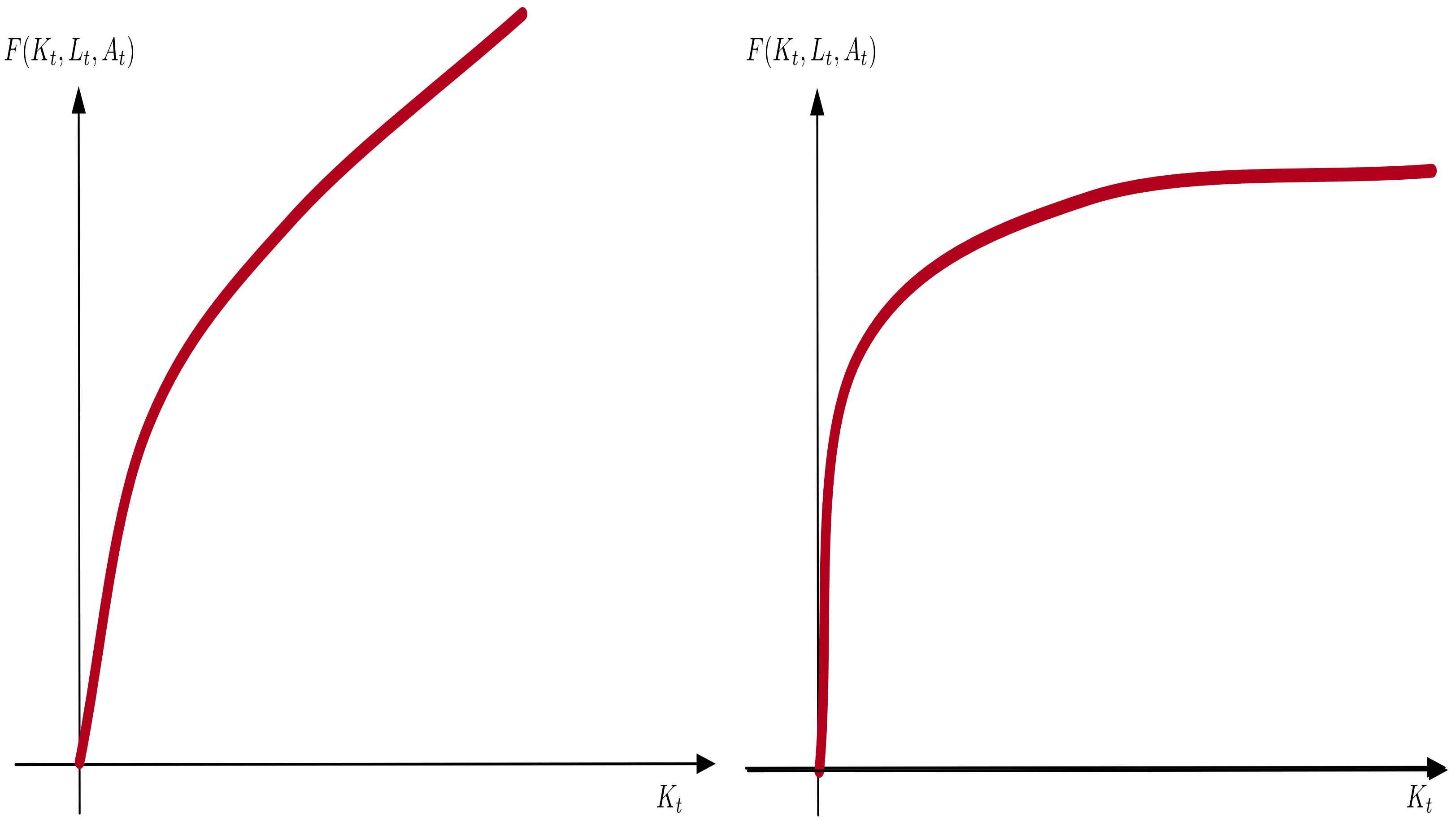
- Assumption 2 (the Inada Conditions). The first derivatives of the aggregate production function have the limits

$$\lim_{K\to 0} F_K(K, L, A) = \infty \quad \text{and} \quad \lim_{K\to\infty} F_K(K, L, A) = 0 \quad \text{all } L > 0$$

$$\lim_{L\to 0} F_L(K, L, A) = \infty \quad \text{and} \quad \lim_{L\to\infty} F_L(K, L, A) = 0 \quad \text{all } K > 0$$

- Remark. Looking ahead, these Inada conditions are *sufficient* for the existence of an interior solution (as we will discuss). Can be relaxed, but for now helps streamline the analysis.

Inada Conditions



Production function on the left does not satisfy the Inada conditions, the marginal product of capital does not decline to zero as capital is accumulated. The production function on the right does satisfy the Inada conditions.

Outline

1. Setup

Aggregate production function

Constant saving rate

2. Special case with constant A and L

Steady state

Transitional dynamics

Comparative statics and the ‘golden rule’

3. Decentralization

Saving and Investment

- Since the economy is closed and there is no government sector

$$S_t = I_t = Y_t - C_t$$

- **Key 'behavioral' assumption of the Solow model**: A constant fraction s of output is saved each period

$$S_t = sY_t, \qquad 0 < s < 1$$

- Hence capital accumulation is given by

$$K_{t+1} - K_t = sY_t - \delta K_t$$

- Using the aggregate production function gives the *law of motion*

$$K_{t+1} = sF(K_t, L_t, A_t) + (1 - \delta)K_t$$

Law of Motion

- Given an initial condition K_0 and exogenous sequences $\{A_t, L_t\}$ we generate an *endogenous* sequence $\{K_t\}$ by iterating on the law of motion

$$K_{t+1} = sF(K_t, L_t, A_t) + (1-\delta)K_t$$

- A *nonlinear* difference equation, but, as we will see, given Assumptions 1 and 2 it has straightforward dynamic properties.

- Once we know K_t we also know output and consumption etc

$$Y_t = F(K_t, L_t, A_t)$$

$$C_t = (1-s)F(K_t, L_t, A_t)$$

- In short, K_t (with exogenous A_t, L_t) summarizes *state* of the economy.

Discussion

- A slightly unusal mix of old-style Keynesian ingredients (exogenous, constant marginal propensity to save) with frictionless production side.

- No 'microfoundations' of savings behavior

  – is s the common savings rate of many identical households? or the aggregate savings rate from many different households?

  – key is that s is exogenous, not that it is constant

  – hard to make welfare statements, how do we rank outcomes?

  – model is silent on such questions

- To begin with, we solve a special case with constant A and L.

Outline

1. **Setup**
   Aggregate production function
   Constant saving rate

2. **Special case with constant A and L**
   Steady state
   Transitional dynamics
   Comparative statics and the 'golden rule'

3. **Decentralization**

Special Case: Constant A and L

- To fix ideas, consider special case with $A_t = A = 1$ and $L_t = L$
- Write model in *per worker* terms

$$y \equiv \frac{Y}{L}, \qquad k \equiv \frac{K}{L}, \ldots \qquad \text{etc}$$

- Using constant returns to scale and $A = 1$ normalization

$$y = \frac{Y}{L} = \frac{F(K, L, 1)}{L} = F\left(\frac{K}{L}, 1, 1\right) = F(k, 1, 1) \equiv f(k)$$

- This is the *intensive form* of the production function

$$y = f(k), \qquad f'(k) > 0, \qquad f''(k) < 0, \qquad f'(0) = \infty, \qquad f'(\infty) = 0$$

- Law of motion for capital per worker is then

$$k_{t+1} = sf(k_t) + (1 - \delta)k_t$$

Law of Motion

- Write the law of motion for capital per worker

$$k_{t+1} = g(k_t) \equiv sf(k_t) + (1-\delta)k_t$$

- An *autonomous* difference equation — no exogenous forcing process.

- Given an initial condition $k_0 > 0$ simply iterate on difference equation

$$\begin{aligned}
k_1 &= g(k_0) \\
k_2 &= g(k_1) = g(g(k_0)) = g^2(k_0) \\
&\vdots \\
k_t &= g(g^{t-1}(k_0)) = g^t(k_0)
\end{aligned}$$

  where $g^t(k)$ denotes the t-th iterate of the function $g(k)$.

- What dynamics for k_t are implied by iterating in this way?

Outline

Steady State

- A *steady state* k^* is a *fixed point* of the law of motion

$$k^* = g(k^*)$$

- So for the Solow model, a steady state k^* is characterized by

$$sf(k^*) = \delta k^*$$

where investment $sf(k^*)$ just offsets depreciation δk^*.

- There is a trivial steady state $k^* = 0$. But we will focus on the *non-trivial* case where $k^* > 0$ is pinned down by

$$\frac{k^*}{y^*} = \frac{k^*}{f(k^*)} = \frac{s}{\delta}$$

Notice that the *capital/output ratio* k^*/y^* is pinned down by s/δ alone, independent of the production function [Swan's approach].

Steady State

- Proposition. Given Assumptions 1 and 2, there is a unique steady state $k^* > 0$ satisfying $k^*/f(k^*) = s/\delta$.

- Proof (Sketch). Assumptions 1 and 2 imply $h(k) \equiv k/f(k)$ is continuous and strictly increasing from $h(0) = 0$ to $h(\infty) = \infty$ so by the *intermediate value theorem* there is a unique $k^* > 0$ satisfying $h(k^*) = s/\delta$.

- Remark. We get *existence* because $h(k)$ is continuous from $(0, \infty)$ to $(0, \infty)$, *uniqueness* because $h(k)$ is strictly increasing.

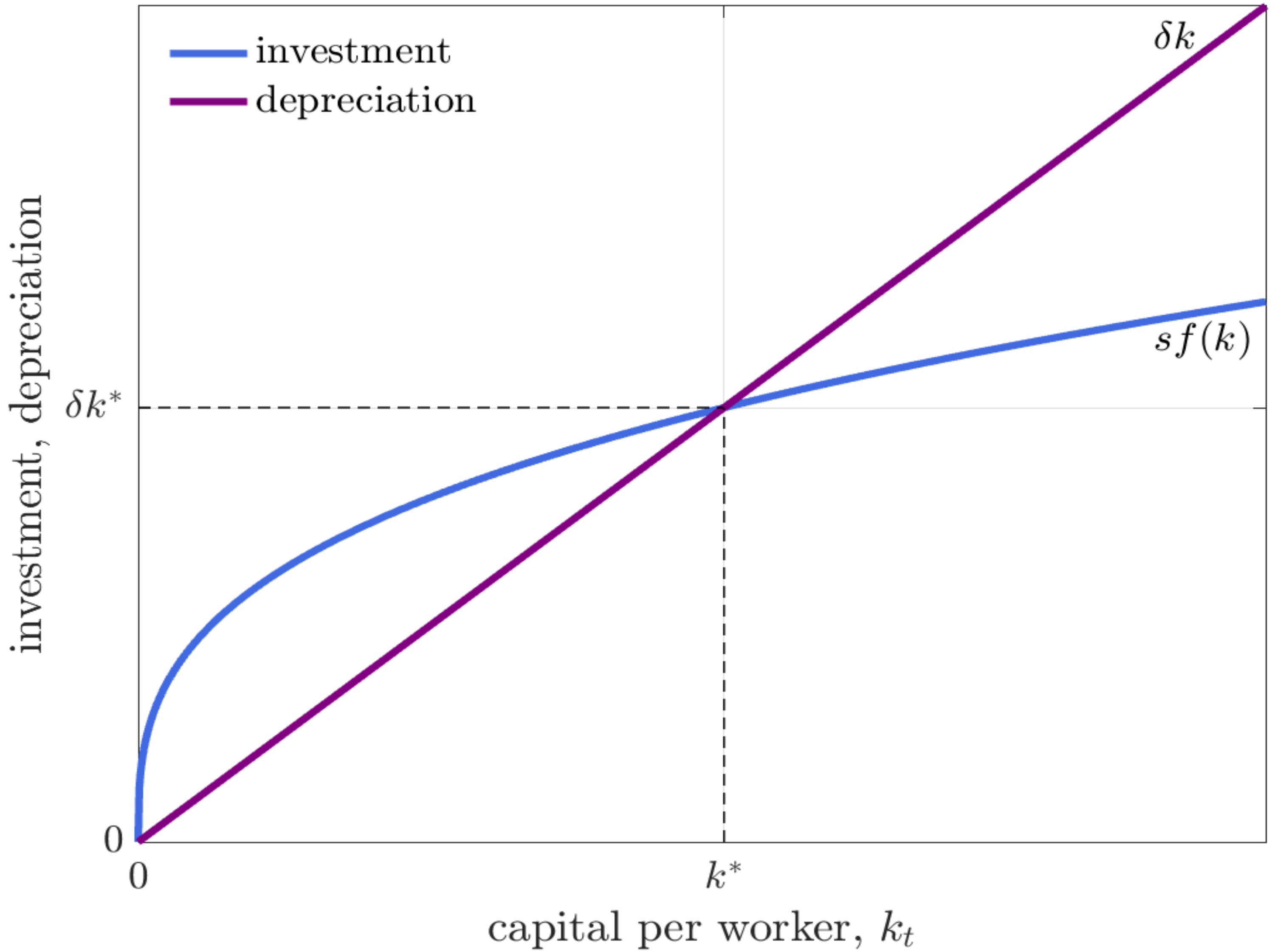
- Given this k^*, steady-state output per worker is simply

$$y^* = f(k^*)$$

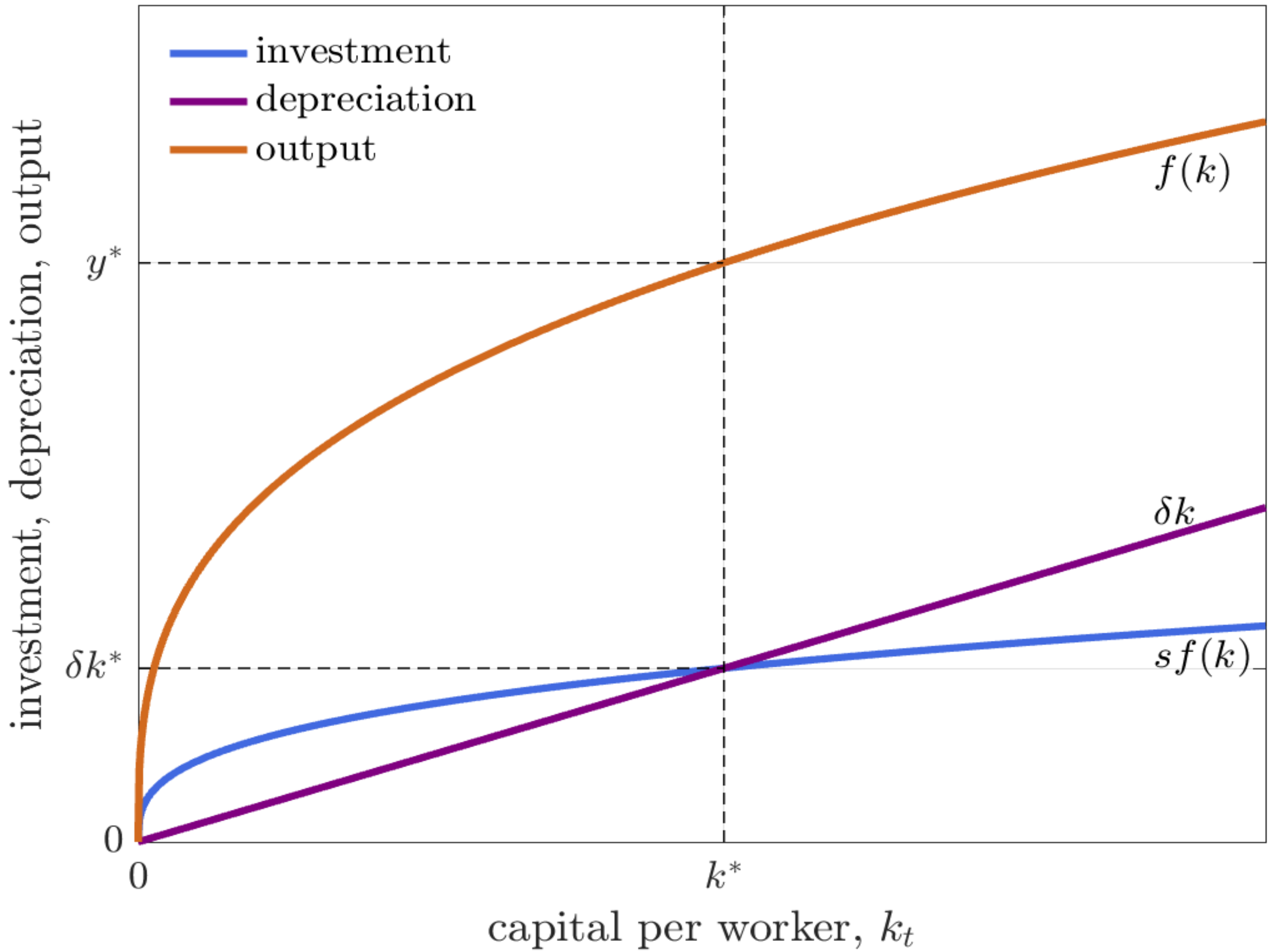
and steady-state consumption per worker is

$$c^* = (1 - s)f(k^*)$$

Solow Diagram



Solow Diagram



Outline

1. Setup

Aggregate production function

Constant saving rate

2. Special case with constant A and L

Steady state

Transitional dynamics

Comparative statics and the ‘golden rule’

3. Decentralization

Transitional Dynamics

- Is this unique steady state k^* *stable*?

- That is, if economy starts at some k_0, does it converge to k^*?

- If so, steady state k^* characterizes the *long-run* outcome for the economy.

- How long is the long run? How *quickly* does k_t converge to k^* if it does?

Global Stability

- PROPOSITION. For any $k_0 > 0$ the economy converges $k_t \to k^*$. Moreover the transition is monotone

$$k_0 < k^* \qquad \Rightarrow \qquad k_0 < k_1 < k_2 < \cdots < k^*$$

$$k_0 > k^* \qquad \Rightarrow \qquad k_0 > k_1 > k_2 > \cdots > k^*$$

Global Stability

- PROPOSITION. For any $k_0 > 0$ the economy converges $k_t \to k^*$. Moreover the transition is monotone

$$k_0 < k^* \qquad \Rightarrow \qquad k_0 < k_1 < k_2 < \cdots < k^*$$

$$k_0 > k^* \qquad \Rightarrow \qquad k_0 > k_1 > k_2 > \cdots > k^*$$

- PROOF (Sketch). First observe $g(k) = sf(k) + (1-\delta)k$ satisfies $g'(k) = sf'(k) + (1-\delta) > 0$ and $g''(k) = sf''(k) < 0$ with unique $k^* > 0$ such that $k^* = g(k^*)$. Hence $g(k) > k$ if and only if $k < k^*$. Therefore $k_{t+1} > k_t$ whenever $k_t \in (0, k^*)$ and moreover

$$k_{t+1} - k^* = g(k_t) - g(k^*) = -\int_{k_t}^{k^*} g'(k)\, dk < 0$$

so the sequence is strictly increasing and bounded above and hence converges to some point, which, by continuity of $g(k)$, is the fixed point k^*. In short, for any $k_0 \in (0, k^*)$ we have $k_t \nearrow k^*$. A symmetric argument establishes that for any $k_0 \in (k^*, \infty)$ we have $k_t \searrow k^*$.

Global Stability

- Remark. Consider the *growth* in capital per worker

$$\gamma(k_t) \equiv \frac{k_{t+1} - k_t}{k_t} = s\frac{f(k_t)}{k_t} - \delta = \frac{s}{h(k_t)} - \delta$$

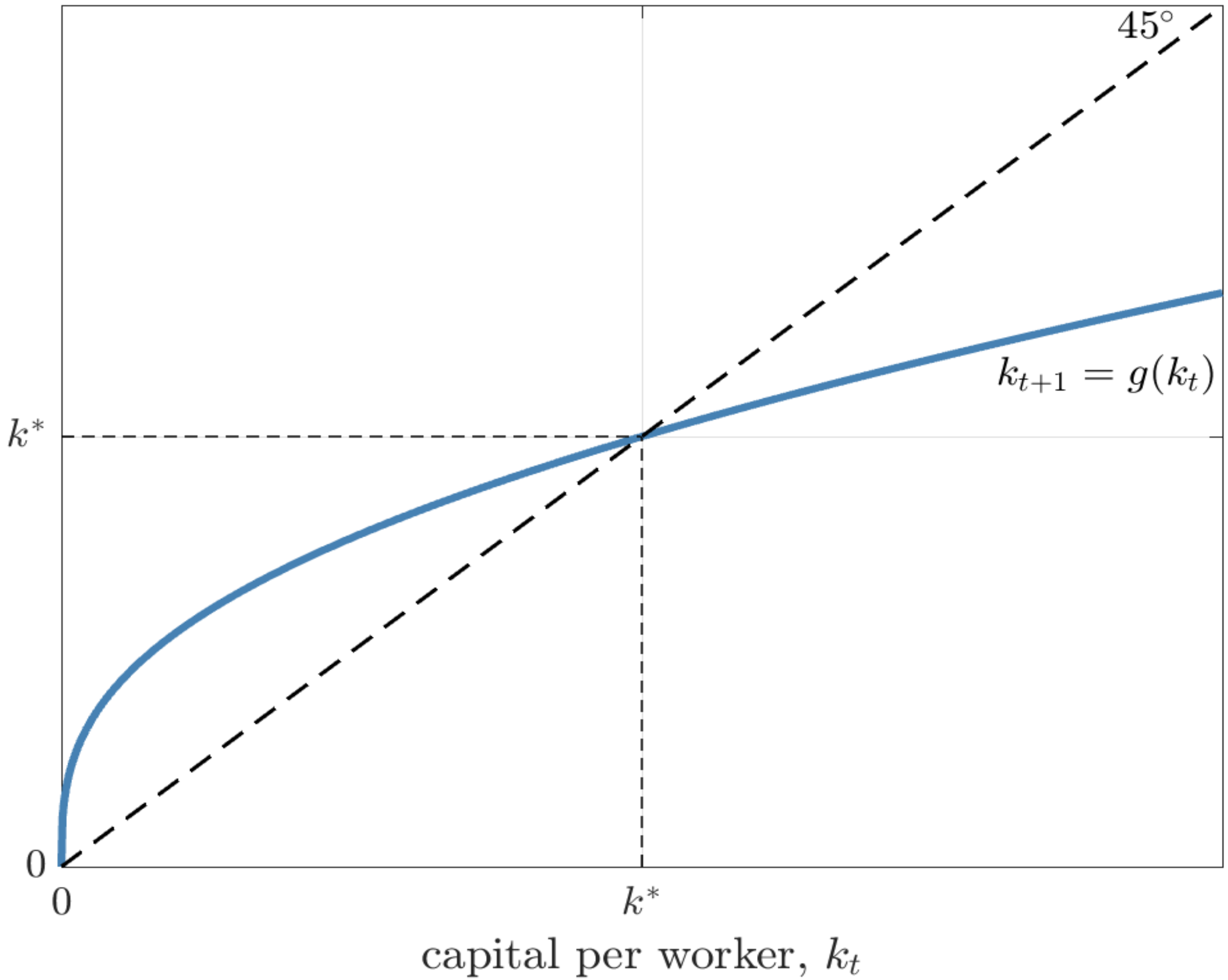
  where as before $h(k_t) = k_t/f(k_t)$ is the capital/output ratio.

- Since $h(k)$ is strictly increasing in k and satisfies $h(k^*) = s/\delta$, the growth rate $\gamma(k)$ is strictly decreasing in k with $\gamma(k) > 0$ iff $k < k^*$.

- So for $k_t < k^*$ the capital stock grows, $\gamma(k_t) > 0$, but at a diminishing rate, with $\gamma(k_t) \searrow 0$ as $k_t \nearrow k^*$.

- And for $k_t > k^*$ the capital stock shrinks, $\gamma(k_t) < 0$, but at a diminishing rate, with $\gamma(k_t) \searrow 0$ as $k_t \searrow k^*$.

Phase Diagram



Speed of Convergence

- How *fast* does $k_t \to k^*$?

- Consider a *linear approximation* to $g(k)$ around k^*

$$k_{t+1} \approx g(k^*) + g'(k^*)(k_t - k^*) = k^* + g'(k^*)(k_t - k^*)$$

- Let $\hat{k}_t \equiv (k_t - k^*)/k^*$ denote the proportional deviation and treat as exact

$$\hat{k}_{t+1} = g'(k^*)\hat{k}_t$$

- Solving this linear difference equation

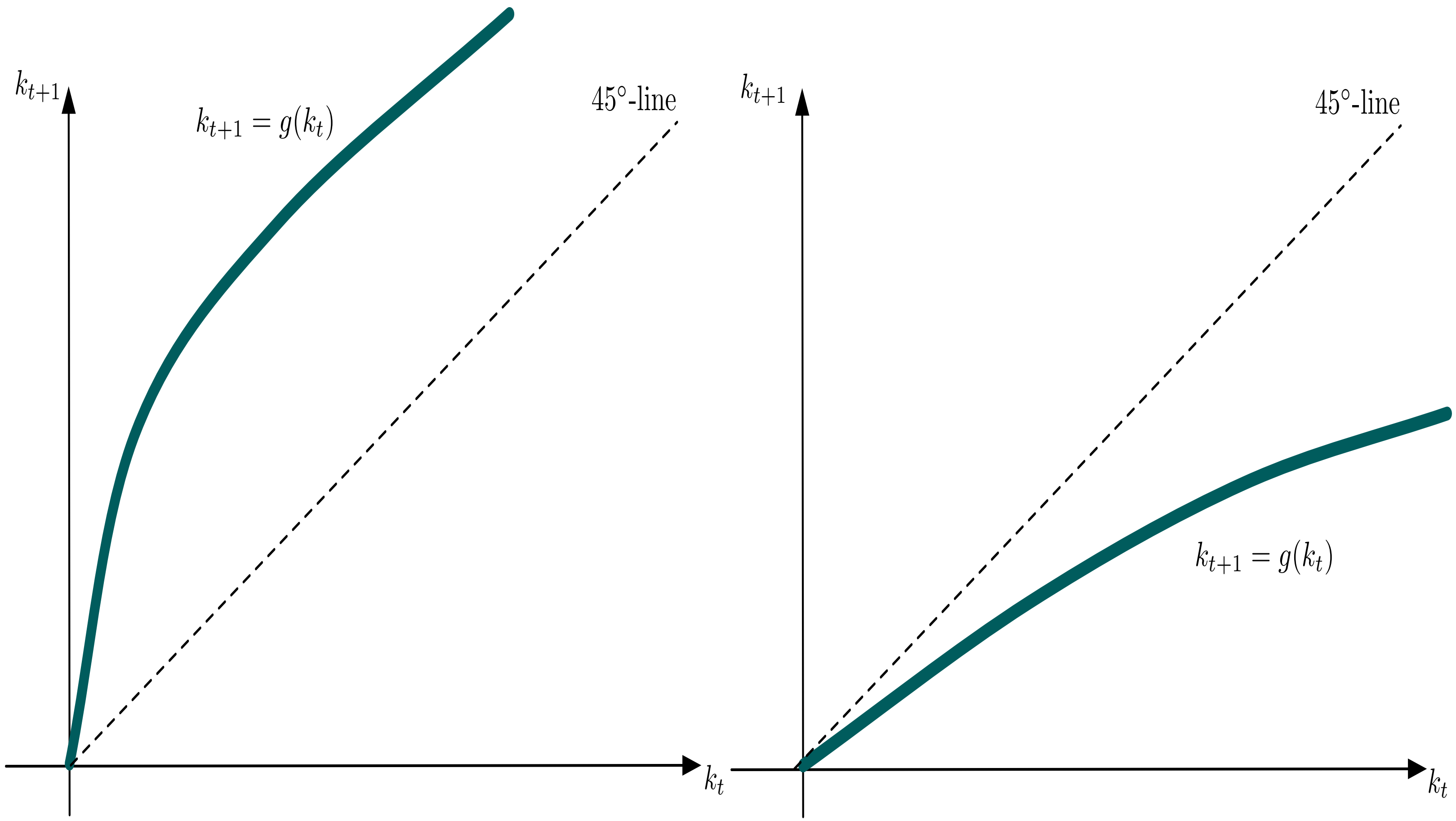
$$\hat{k}_t = g'(k^*)^t\,\hat{k}_0, \qquad g'(k^*) \in (0,1)$$

- Converges quickly if $g'(k^*)$ close to 0, converges slowly if $g'(k^*)$ close to 1. *Half-life* of deviations

$$t^* = \frac{\log(1/2)}{\log g'(k^*)} > 0$$

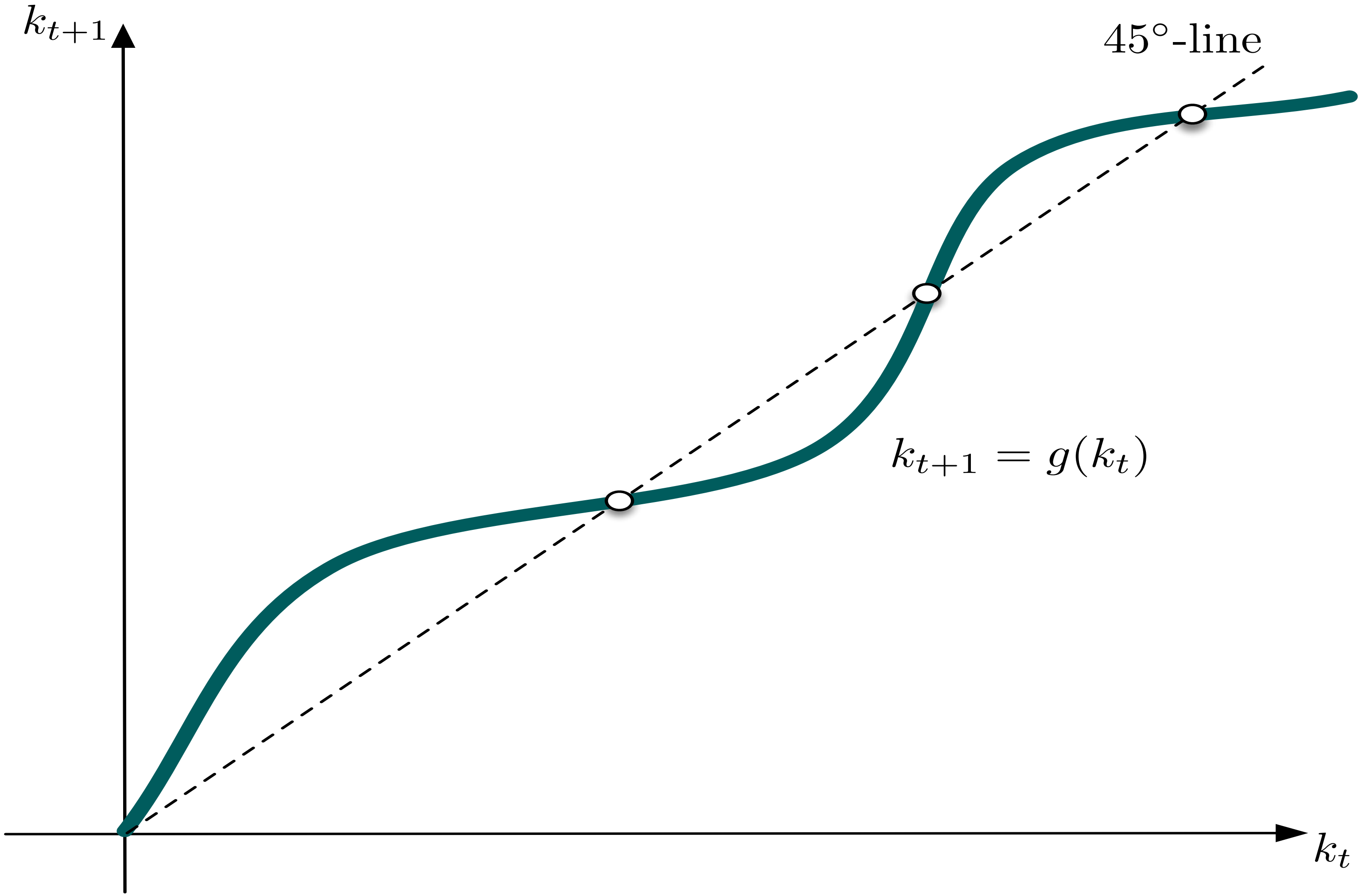
where $g'(k^*)$ pinned down by parameters $s, \delta, f'(k)$ — for what values of these parameters is $g'(k^*)$ close to 1?

Non-Existence of Steady State



Curvature in the law of motion $g(k)$ inherited from curvature in $f(k)$. Failure to satisfy the Inada conditions *may* imply non-existence of a non-trivial steady state $k^* > 0$. For the law of motion on the left, the capital stock grows unboundedly. For the law of motion on the right the capital stock shrinks to zero (the trivial steady state).

Non-Uniqueness of Steady State



Non-monotonicity of $g(k)$ *may* imply multiple non-trivial steady states. If so, some will be locally stable while others are unstable.

Outline

1. Setup
- Aggregate production function
- Constant saving rate

2. Special case with constant A and L
- Steady state
- Transitional dynamics
- Comparative statics and the 'golden rule'

3. Decentralization

Comparative Statics

- Steady state condition pins down k^* in terms of parameters

$$sf(k^*) = \delta k^*$$

- Since $h(k) = k/f(k)$ is strictly increasing, k^* is strictly increasing in s/δ.

- Since $f(k)$ is strictly increasing, $y^* = f(k^*)$ is strictly increasing in s/δ.

- But steady state c^* is *non-monotone* in s. For intuition, consider the extremes $s = 0$ and $s = 1$.

- What saving rate *maximizes* steady-state consumption per worker?

$$c^*(s) = (1 - s)f(k^*(s)) = f(k^*(s)) - \delta k^*(s)$$

- Saving rate which achieves the max known as the *golden rule* saving rate.

Golden Rule

- PROPOSITION. Steady-state consumption per worker is maximized by the saving rate s^*_{GR} such that $k^*_{\mathrm{GR}} \equiv k^*(s^*_{\mathrm{GR}})$ satisfies

$$f'(k^*_{\mathrm{GR}}) = \delta$$

- PROOF (Sketch). The first order condition is

$$\frac{dc^*(s)}{ds} = [f'(k^*(s)) - \delta]\,\frac{dk^*(s)}{ds} = 0$$

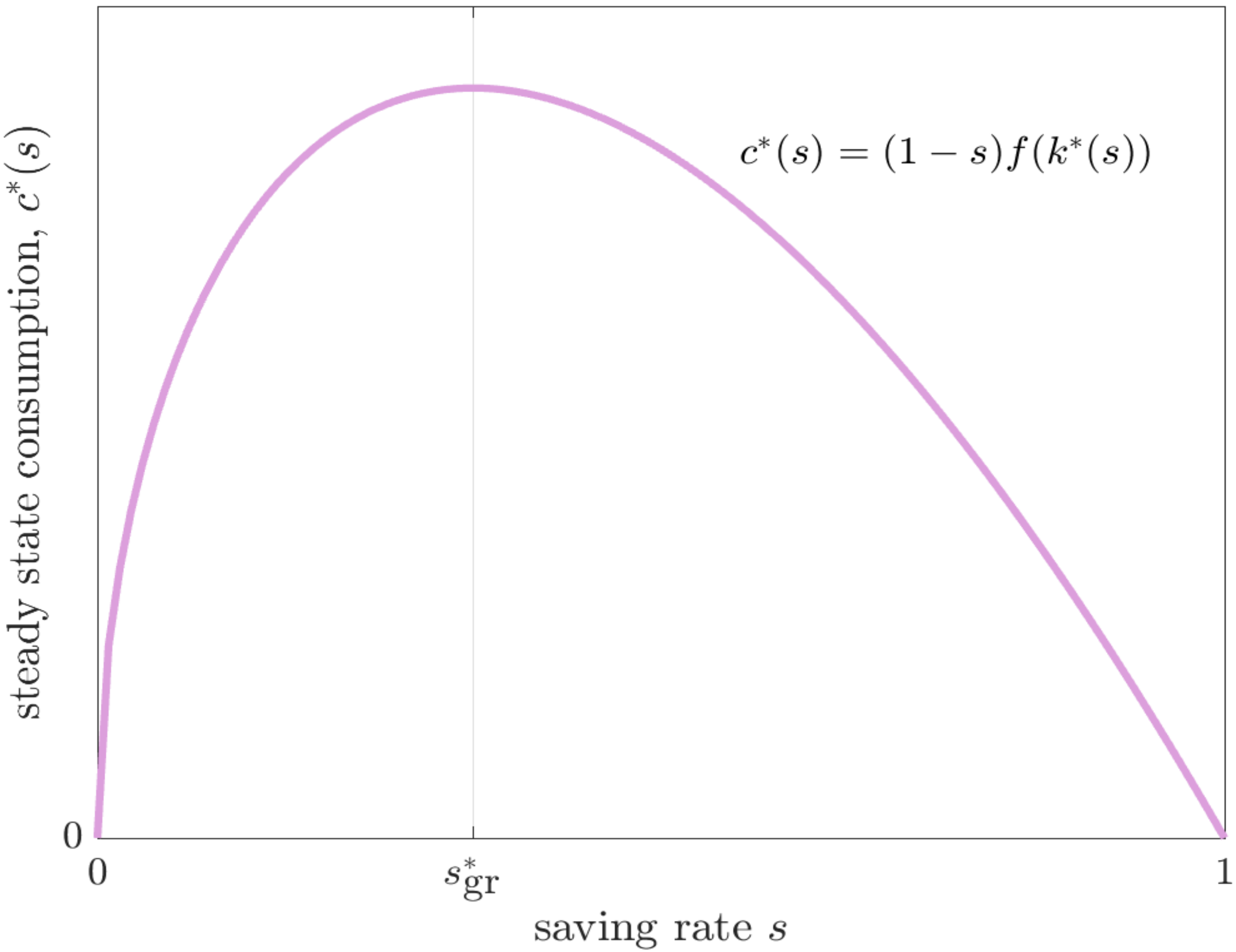
Since $k^*(s)$ is strictly increasing in s, there is a unique critical point s^*_{GR} such that $f'(k^*_{\mathrm{GR}}) = \delta$. Moreover, since $f''(k) < 0$ we know $f'(k^*(s)) > \delta$ if and only if $s < s^*_{\mathrm{GR}}$ hence this critical point maximizes $c^*(s)$.

- REMARK. Using the steady state condition $sf(k^*) = \delta k^*$ we can also write

$$s^*_{\mathrm{GR}} = \frac{f'(k^*_{\mathrm{GR}})k^*_{\mathrm{GR}}}{f(k^*_{\mathrm{GR}})}$$

At the golden rule, the saving rate equals the elasticity of output with respect to capital — e.g., $s^*_{\mathrm{GR}} = \alpha$ if $f(k) = k^\alpha$ for $\alpha \in (0,1)$.

Golden Rule



$c^*(s) = (1 - s)f(k^*(s))$
steady state consumption, $c^*(s)$
0
0
s^*_{gr}
1
saving rate s

Dynamic Inefficiency

- When s is such that $k^* < k^*_{\text{GR}}$, an increase in s will *increase* steady state consumption per worker.

- When s is such that $k^* > k^*_{\text{GR}}$, an increase in s will *decrease* steady state consumption per worker.

- In latter case, steady-state consumption can be increased by *saving less*.

- In a sense, there is 'too much capital,' a form of '*dynamic inefficiency*'.

- But strictly speaking, we can't say that unless we specify a way of ranking outcomes [is more consumption always better?]

- Will return to this question in coming classes.

Outline

Decentralization

- Let the final good be the numeraire.

- *Representative household*

  - L identical households
  - each endowed with 1 unit of labor per period and initial capital k_0
  - take wage w_t and rental rate of capital R_t as given
  - supply labor and capital to firms
  - save fraction s of their income $w_t + R_t k_t$, zero economic profits
  - since capital depreciates at rate δ, net *return* on their capital is $r_t = R_t - \delta$

- *Representative firm*

  - identical technology $Y = F(K, L, A)$
  - indeterminate number of firms, since constant returns to scale
  - take wage w_t and rental rate of capital R_t as given
  - choose labor and capital demand to max profits
  - owned by households, but constant returns to scale + competitive factor prices $\Rightarrow$ zero economic profits so nothing left to distribute to owners

Representative Firm

- Taking w and R as given choose L and K to max profits

$$F(K, L, A) - wL - RK$$

- Note final good is numeraire, relative price of output is 1.

- *Static problem*, no adjustments costs etc on factors of production.

- Written in aggregate variables since individual firm size is indeterminate.

- Concave problem so first order conditions necessary and sufficient

$$F_K(K, L, A) = R$$
$$F_L(K, L, A) = w$$

Determines factor demands K, L as functions of factor prices w, R.

Constant Returns to Scale

- Because F has constant returns to (is *homogeneous of degree one* in) K, L and factors are priced competitively, there are zero economic profits

$$F(K, L, A) - wL - RK = F(K, L, A) - F_L(K, L, A)L - F_K(K, L, A)K = 0$$

- Also using Euler's theorem on homogeneous functions, the marginal products are *homogeneous of degree zero* so that we can write

$$\begin{aligned} F_K(K, L, A) &= F_K(k, 1, A) \\ F_L(K, L, A) &= F_L(k, 1, A) \end{aligned}$$

- The factor demands pin down the $k = K/L$ ratio in terms of the factor price ratio w/R but the actual *scale* of production is indeterminate.

- Using the intensive form $f(k) \equiv F(k, 1, A)$ of the production function

$$R = f'(k) = F_K(k, 1, A) > 0$$

and

$$w = f(k) - f'(k)k = F_L(k, 1, A) > 0$$

Market Clearing

- Three markets with two relative prices

  – factor market for labor (relative price w)
  – factor market for capital (relative price R)
  – market for final output (numeraire)

- By *Walras' law*, if markets for labor and capital clear so does market for final output, i.e., one market clearing condition is redundant.

- We almost have something that looks like a competitive equilibrium, except no household optimization.

Next class

- Solow model in *continuous time*, slightly cleaner.

- A first look at sustained growth, without diminishing returns to capital.

- Balanced growth, and different types of technological change.

Homework

- Check. Given Assumptions 1 and 2, show that the capital/output ratio $h(k) = k/f(k)$ is continuous and strictly increasing from $(0, \infty)$ to $(0, \infty)$.

  Hint: use l'Hôpital's rule.

- Check. Given Assumptions 1 and 2, show that $g'(k^*) \in (0, 1)$.