Economic Growth

Lecture 10: Endogenous growth, part two

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Endogenous Growth

- Now turn to models where improvements in productivity are the result of deliberate investments in R&D-like activities.

- Main idea: investments in R&D change the set of inputs used in production. Two general approaches:

  (i) '*horizontal*' greater *variety* of inputs used in production

  [new product varieties, increased specialization, i.e., 'division of labor']

  (ii) '*vertical*' improvements in *quality* of existing varieties

  [better product varieties]

- Focus today on horizontal approach, will later return to vertical approach.

- Requires departing from perfect competition. Throughout we use the Dixit-Stiglitz (1977) model of *monopolistic competition* with CES demand.

Innovation

- Focus on role of *innovation* in generating new *ideas*, 'blueprints' for production. Romer (1990) highlights three important factors:

  (i) ideas are *nonrival*, my use of an idea does not preclude you using it

  (ii) *increasing returns to scale*, constant returns to capital and labor, but ideas are also produced and hence increasing returns for the whole economy

  (iii) *monopoly profits* as return to sunk/fixed costs of investment in research

- Will see how these ingredients interact in two models

  – 'lab equipment model', new products created using final output
  [Romer 1987, linear growth model in equilibrium]

  – 'knowledge spillovers model' – new products created using scarce factors
  [Romer 1990, need spillovers from past R&D to get endogenous growth]

Outline

Lab Equipment Model

- All that is required for research is the equipment, i.e., the *lab*.

- In particular, research can be produced using final output.

  – perfectly competitive *final good producers*, numeraire

    [produce using labor and intermediates]

  – monopolistically competitive *intermediate good producers*

    [produce using final good]

- Do not need actual researchers, i.e., labor, or other scarce factors.

- Minimizes the roll of spillovers and externalities.

Outline

Representative Household

- Representative household of constant size L.

- Balanced growth preferences, inelastic labor supply

$$U = \int_0^\infty e^{-\rho t} \left(\frac{c(t)^{1-\theta} - 1}{1-\theta} \right) dt$$

- Consumption per worker $c(t) \equiv C(t)/L$.

- Owns *balanced portfolio* of all the firms in the economy.

Final Good Producers

- Perfectly competitive final good producers.
- Use labor L and a continuum of intermediate goods $v \in [0, N(t)]$.
- Production function for final good

$$Y(t) = A\, X(t)^{1-\beta}\, L^{\beta}, \qquad 0 < \beta < 1, \qquad A > 0$$

where $X(t)$ is a CES aggregate of the bundle of intermediates

$$X(t) = \left(\int_0^{N(t)} x(v,t)^{1-\beta}\, dv \right)^{\frac{1}{1-\beta}}$$

where $N(t) > 0$ is number of intermediate varieties, to be determined.

- Constant returns to scale given $N(t)$, taken as given.
- Think of $x(v,t)$ as lab equipment that depreciates fully after use.

Final Good Producers

- Let $p(v,t)$ denote the price of intermediate v in units of the final good.

- Let $P(t)$ denote the *price index* for the bundle of intermediates, satisfies

$$P(t)X(t) = \int_0^{N(t)} p(v,t)x(v,t)\,dv$$

- Break final good producer problem into two steps:

  (i) taking $P(t), w(t)$ as given choose aggregate $X(t)$ and $L(t)$ to max profits

  (ii) taking $p(v,t)$ as given choose individual $x(v,t)$ to min cost $P(t)X(t)$

- **Step (i).** First order conditions for aggregate $X(t)$ and $L(t)$ are standard

$$\begin{aligned}(1-\beta)Y(t) &= P(t)X(t)\\ \beta Y(t) &= w(t)L\end{aligned}$$

Final Good Producers

- **Step (ii).** Choose $x(v,t)$ to minimize

$$P(t)X(t) = \int_0^{N(t)} p(v,t)x(v,t)\,dv$$

subject to

$$X(t) = \left(\int_0^{N(t)} x(v,t)^{1-\beta}\,dv\right)^{\frac{1}{1-\beta}}$$

- Implies demand curve for each intermediate

$$x(v,t) = \left(\frac{p(v,t)}{P(t)}\right)^{-\frac{1}{\beta}} X(t)$$

and hence price index

$$P(t) = \left(\int_0^{N(t)} p(v,t)^{1-\frac{1}{\beta}}\,dv\right)^{\frac{1}{1-\frac{1}{\beta}}}$$

- Constant elasticity demand for intermediates with elasticity $1/\beta > 1$.

Final Good Producers

- For future reference, note that we can write the optimality conditions for the individual $x(v,t)$ and aggregate $X(t)$ in parallel fashion

$$(1-\beta)Ax(v,t)^{-\beta}L^{\beta} = p(v,t)$$

$$(1-\beta)AX(t)^{-\beta}L^{\beta} = P(t)$$

(taking the ratio of these recovers the demand curve)

Intermediate Good Producers

- A firm owning the blueprint to an intermediate good of type v has a monopoly on that good.

- Produce quantity $x(v,t)$ of their intermediate at constant marginal cost $\psi > 0$ *units of the numeraire final good.*

- Market power, internalize the demand curve from final good producers.

- Static profits

$$\pi(v,t) \equiv (p(v,t) - \psi)\, x(v,t), \qquad x(v,t) = \left(\frac{p(v,t)}{P(t)}\right)^{-\frac{1}{\beta}} X(t)$$

- Choose price to maximize profits, giving

$$p(v,t) = \frac{1}{1-\beta}\,\psi$$

(a markup $1/(1-\beta) > 1$ over marginal cost ψ).

Intermediate Good Producers

- With this constant markup, static profits are proportional to revenues and hence to output

$$\pi(v,t) = (p(v,t) - \psi)x(v,t) = \beta p(v,t)x(v,t) = \frac{\beta}{1-\beta}\,\psi\, x(v,t)$$

- Intertemporal profits of an intermediate good producer at time t

$$V(v,t) \equiv \int_t^{\infty} \exp\Big(-\int_t^s r(s')\,ds'\Big)\,\pi(v,s)\,ds$$

- Market value of a claim to the stream of profits $\pi(v,s)$ for $s \geq t$.
- Equivalently, in flow terms

$$\dot{V}(v,t) = -\pi(v,t) + r(t)V(v,t) \qquad \Leftrightarrow \qquad r(t)V(v,t) = \pi(v,t) + \dot{V}(v,t)$$

- Market value $V(v,t)$ may change either because of changes to flow profits $\pi(v,t)$ or changes in real interest rate $r(t)$.

Investment in R&D

- There is *free entry* into research. Anyone can use one unit of the final good to produce new blueprints with *arrival rate* $\eta > 0$.

- Successful innovation results in a perpetual patent, market value $V(v,t)$.

- Let $Z(t)$ denote total units of the final good used for R&D purposes.

- Aggregate number of varieties evolves according to

$$\dot{N}(t) = \eta Z(t), \qquad N(0) > 0 \quad \text{given}$$

- Market clearing condition for the numeraire final good is then

$$C(t) + \psi \int_0^{N(t)} x(v,t)\, dv + Z(t) = Y(t)$$

- Remark. At the individual level, innovation is a *Poisson process* with arrival rate η. But a law of large numbers applies and there is no aggregate uncertainty.

Free Entry Condition

- Suppose the market value satisfies $\eta V(v,t) > 1$. Then strictly positive profits can be made by investing in the production of new blueprints.

- Free entry requires the complementary slackness conditions

$$\eta V(v,t) \leq 1$$

and

$$\Big[\eta V(v,t) - 1\Big] Z(t) = 0$$

- With positive entry, $Z(t) > 0$, we will have

$$V(v,t) = \frac{1}{\eta}$$

Representative Household

- Let $\mathcal{A}(t) \equiv \int_0^{N(t)} V(v,t)\,dv$ denote market value of household's portfolio.
- Let $a(t) \equiv \mathcal{A}(t)/L$ denote market value per worker.
- Flow constraint

$$\dot{a}(t) = r(t)a(t) + w(t) - c(t)$$

- Consumption Euler equation

$$\frac{\dot{c}(t)}{c(t)} = \frac{r(t) - \rho}{\theta}$$

- Transversality condition

$$\lim_{T\to\infty} \exp\left(-\int_0^T r(s)\,ds\right)a(T) = 0$$

Outline

Solving the Model

- Price of each intermediate, constant

$$p(v,t) = \frac{1}{1-\beta}\,\psi \equiv \bar{p}$$

- The quantity of each intermediate is then pinned down by the demand from final good producers, constant and proportional to L

$$\begin{aligned} (1-\beta)Ax(v,t)^{-\beta}L^{\beta} &= p(v,t) \\ &\Rightarrow \\ x(v,t) &= \left((1-\beta)^2\,\frac{A}{\psi}\right)^{1/\beta} L \equiv \bar{x}\,L \end{aligned}$$

- Static profits from intermediate also constant and proportional to L

$$\pi(v,t) = (p(v,t)-\psi)x(v,t) = \beta\bar{p}\,x(v,t) = \beta\bar{p}\bar{x}\,L \equiv \bar{\pi}\,L$$

Aggregate Production Function

- Aggregate quantity of intermediates is then

$$X(t) = \left(\int_0^{N(t)} x(v,t)^{1-\beta}\, dv \right)^{\frac{1}{1-\beta}} = N(t)^{\frac{1}{1-\beta}}\, \bar{x}\, L$$

  increasing in $N(t)$ with elasticity $1/(1-\beta) > 1$, i.e., the markup.

- Aggregate production function is then

$$Y(t) = A\, X(t)^{1-\beta} L^{\beta} = A\bar{x}^{1-\beta}\, N(t) L$$

- Remarks. Constant returns to the sole factor of production L, but *increasing returns* to $N(t)$ and L together.

  Not an 'AK' model but an 'AN' model. Output per worker, $Y(t)/L$, proportional to $N(t)$.

Gains from Variety

- Relative size of each intermediate

$$\frac{x(v,t)}{X(t)} = N(t)^{-\frac{1}{1-\beta}}$$

- Average productivity is higher when there are more varieties but each variety uses a smaller share of scarce resources (staving off diminishing marginal productivity to each individual variety).

- Notice also

$$P(t) = \left(\int_0^{N(t)} p(v,t)^{1-\frac{1}{\beta}}\,dv\right)^{\frac{1}{1-\frac{1}{\beta}}} = N(t)^{\frac{1}{1-\frac{1}{\beta}}}\,\bar{p}$$

so price of intermediate bundle is decreasing in $N(t)$.

- Remark. Sometimes referred to as a '*love of variety*' effect [from the consumer version of this problem].

Equilibrium Dynamics

- To streamline the exposition, suppose parameters are such that there is entry in equilibrium

$$Z(t) > 0$$

- Then free-entry condition implies

$$V(v,t) = \frac{1}{\eta} \qquad \Rightarrow \qquad \mathcal{A}(t) = \frac{N(t)}{\eta}$$

- Hence, from the flow version of intertemporal profits, real interest rate is

$$r(t) = \frac{\pi(v,t) + \dot{V}(v,t)}{V(v,t)} = \eta\bar{\pi}L \equiv r$$

- Growth in aggregate consumption $C(t) = c(t)L$ is then given by

$$\frac{\dot{C}(t)}{C(t)} = \frac{r - \rho}{\theta} \equiv g$$

Equilibrium Dynamics

- To pin down level of consumption we need to couple Euler equation with the law of motion for new varieties

$$\dot{N}(t) = \eta Z(t) = \eta\Big(Y(t) - \psi \int_0^{N(t)} x(v,t)\,dv - C(t)\Big)$$

$$= \eta\Big(\underbrace{A\bar{x}^{1-\beta} - \psi\bar{x}}_{\text{proportional to } \bar{\pi}}\Big)L\,N(t) - \eta C(t)$$

$$= \eta b\bar{\pi} L\,N(t) - \eta C(t)$$

- The last line uses our previous calculation that $\bar{\pi}$ is proportional to $\bar{x}$ and where the constant of proportionality $b > 1$ is given by

$$A\bar{x}^{1-\beta} - \psi\bar{x} = \left(\frac{1}{(1-\beta)^2} - 1\right)\psi\bar{x} = \left(\frac{2-\beta}{1-\beta}\right)\bar{\pi} \equiv b\bar{\pi} > 0$$

Equilibrium Dynamics

- Then recalling $r \equiv \eta\bar{\pi}L$ we can write the system

$$\dot{N}(t) = rbN(t) - \eta C(t), \qquad \dot{C}(t) = gC(t)$$

- Same structure as the 'AK' model from Lecture 9.

- Write as second-order differential equation in $N(t)$

$$\ddot{N}(t) = (rb + g)\dot{N}(t) - rbgN(t)$$

- Roots given by

$$\lambda_1 = g, \qquad \lambda_2 = rb$$

Equilibrium Dynamics

- Choose $\lambda_1 = g$. Then $\dot{N}(t) = gN(t)$ and from the law of motion for new varieties we have the policy function

$$C(t) = \frac{rb - g}{\eta} N(t)$$

- To satisfy the transversality condition we need

$$\lim_{t \to \infty} e^{-rt} e^{gt} N(0) = 0 \qquad \Leftrightarrow \qquad r > g$$

- Assumptions. To streamline the exposition, suppose parameters satisfy

  (i) $\eta \bar{\pi} L > \rho$, so economy is growing, $g > 0$
  (ii) $\rho > (1 - \theta)\eta \bar{\pi} L$, so TVC is satisfied, $r > g$

- Remarks. Since $b > 1$, if the transversality condition is satisfied then the policy function $C = (rb - g)N/\eta$ is increasing in N. If the transversality condition is satisfied, $r > g$, and the economy is growing $g > 0$, we must also have positive entry.

Balanced Growth Path

- For this configuration of parameters we have a balanced growth path

$$\frac{\dot{N}(t)}{N(t)} = \frac{\dot{Y}(t)}{Y(t)} = \frac{\dot{C}(t)}{C(t)} = g > 0$$

where

$$g = \frac{\eta \bar{\pi} L - \rho}{\theta}, \qquad \bar{\pi} \equiv \frac{\beta}{1-\beta} \psi \underbrace{\left((1-\beta)^2 \frac{A}{\psi} \right)^{1/\beta}}_{=\bar{x}}$$

- Expenditure on intermediates $P(t)X(t)$ grows at rate g with

$$\frac{\dot{X}(t)}{X(t)} = \frac{1}{1-\beta} g > 0, \qquad \text{and} \qquad \frac{\dot{P}(t)}{P(t)} = \frac{1}{1-\frac{1}{\beta}} g < 0$$

- Real wage $w(t)$ also grows at rate g.
- The composite parameter $\bar{\pi}$ is strictly increasing in the level of A and strictly decreasing in the marginal cost of producing intermediates ψ.

No Entry [For Completeness]

- Alternatively, if $\eta\bar{\pi}L < \rho$, then $V(v, 0) < 1/\eta$ hence *no entry*, $Z(0) = 0$.

- *If no entry, then no growth,* $g = 0$. Number of intermediates stays constant at the initial $N(0) > 0$.

- Output constant at

$$Y(0) = A\bar{x}^{1-\beta}N(0)L$$

- Consumption constant at

$$C(0) = Y(0) - \psi\bar{x}N(0)L = (A\bar{x}^{1-\beta} - \psi\bar{x})N(0)L = b\bar{\pi}N(0)L$$

supported by real interest rate $r = \rho > \eta\bar{\pi}L$.

Discussion

- *Scale effects.* Growth rate in increasing in size L.

- *No transitional dynamics.* If at any time t we have positive entry, $Z(t) > 0$, then immediately $V(v,t) = 1/\eta$, so have a constant real interest rate $r > \rho$ and the economy is on the balanced growth path.

  In particular, either the economy is conducive to growth and there is positive entry at time $t = 0$ or not.

- *Relationship to AK model.* Linear growth, similar mathematical structure. But economics is a bit different.

Equilibrium is Inefficient

- Unlike the basic AK model with perfect competition, here with monopolistic competition the equilibrium is inefficient.

- There are two sources of inefficiency [market failure]

  (i) *intensive margin*, intermediate producers have market power, markup over marginal cost; at any point in time, output is inefficiently low

  (ii) *extensive margin*, while successful innovators earn monopoly profits, social return to innovation exceeds private return; $N(t)$ inefficiently low

- To see these more precisely, need to solve the planning problem.

Outline

Planning Problem

- Break planning problem into two steps:

  (i) taking $N(t)$ as given, solve static allocation problem

  (ii) given static allocation, solve $N(t)$ dynamic problem

- **Step (i).** Given N, planner's static allocation problem is to choose intermediates $x(v)$ to maximize final output, net of the cost of producing intermediates out of final goods,

$$AL^{\beta} \int_0^N x(v)^{1-\beta}\, dv - \psi \int_0^N x(v)\, dv$$

- The solution is to produce a constant amount of each variety

$$x^*(v) = \left((1-\beta)\,\frac{A}{\psi}\right)^{1/\beta} L \equiv \bar{x}^*\, L$$

- Larger than decentralized equilibrium counterpart.

Planning Problem

- Plugging this back into the objective we get output (net of the cost of producing intermediates) as a function of N, which we can write

$$F(N) = (A\bar{x}^{*\,1-\beta} - \psi\bar{x}^*)NL \equiv \bar{\pi}^* NL$$

- **Step (ii).** The planner's dynamic problem is then to maximize

$$U = \int_0^\infty e^{-\rho t} \left(\frac{C(t)^{1-\theta} - 1}{1-\theta} \right) dt$$

subject to the law of motion for new varieties

$$\dot{N}(t) = \eta(F(N(t)) - C(t)) = \eta(\bar{\pi}^* LN(t) - C(t))$$

- Remark. Just as $\bar{\pi}$ controls the incentives to enter in the decentralized equilibrium counterpart problem, now $\bar{\pi}^*$ controls the planner's incentives to create new varieties.

Planning Problem

- The planner's consumption Euler equation is then

$$\frac{\dot{C}^*(t)}{C^*(t)} = \frac{\eta F'(N^*(t)) - \rho}{\theta} = \frac{\eta \bar{\pi}^* L - \rho}{\theta}$$

- So the planner chooses constant growth

$$g^* = \frac{\eta \bar{\pi}^* L - \rho}{\theta}$$

- By comparison in the decentralized equilibrium

$$g = \frac{\eta \bar{\pi} L - \rho}{\theta}$$

- Thus we have

$$g^* > g \qquad \Leftrightarrow \qquad \bar{\pi}^* > \bar{\pi}$$

- Proposition. Unambiguously, the planner chooses $g^* > g$.

Planning Problem

- PROOF (Sketch). Consider the *surplus* created by each new variety

$$s(x) \equiv Ax^{1-\beta} - \psi x$$

The function $s(x)$ is strictly concave with unique interior maximum at

$$\bar{x}^* \equiv \left((1-\beta)\frac{A}{\psi}\right)^{1/\beta} > \left((1-\beta)^2\frac{A}{\psi}\right)^{1/\beta} \equiv \bar{x}$$

Hence we conclude

$$\bar{\pi}^* = s(\bar{x}^*) = \max_{x \geq 0} s(x) > s(\bar{x}) = \bar{\pi}$$

Hence $g^* > g$.

- REMARKS. The planner uses each new variety $N(t)$ more intensively, and hence values each new variety more than they are valued in the decentralized equilibrium. Planner extracts maximum surplus from each variety and so social return to new varieties is greater than private return.

Outline

Implementing the Efficient Allocation

- How can planner's solution be implemented in decentralized equilibrium?

- Two natural possibilities

  (i) *subsidies to research*, i.e., direct incentives to create new varieties

  (ii) *subsidies to intermediate inputs*, i.e., using subsidies to induce intermediate good producers to operate at the efficient level

  [offsetting the static monopoly distortion]

- REMARKS.

  – but notice that if the static monopoly distortion is corrected, so that the returns per variety are given by $\bar{\pi}^*$ instead of $\bar{\pi}$, then the dynamic distortion is *also* corrected, $g = g^*$

  – implicitly such subsidies must be funded by non-distortionary (lump-sum) taxes; more generally we should be thinking of the costs/benefits of different configurations of distortions

Effects of Competition

- Suppose there is a large *competitive fringe* that can copy the blueprint of any incumbent monopolist producer of an intermediate.

- Suppose also that such copy-cats have marginal cost of production $\gamma\psi > \psi$, greater than that of the incumbent monopolist.

- If cost-disadvantage $\gamma > 1$ is larger than benchmark markup $1/(1-\beta)$, the competitive fringe is no threat to the incumbent.

- But if $\gamma < 1/(1-\beta)$ the fringe can undercut the incumbent monopolist, leading to the '*limit-pricing*' outcome

$$p = \gamma\psi < \frac{1}{1-\beta}\psi$$

- Since this is less than profit-maximizing price, profits are lower hence the incentives to enter are lower and the growth rate of the economy is lower.

Outline

Knowledge Spillovers

- In lab equipment model, research uses final output which has a linear accumulation technology AN much as in a competitive AK model.

- Now turn to an alternative model where research uses *scarce* factors of production, the *labor* of researchers, which is not linearly reproducible.

- In this setting, there can not be endogenous growth unless there are spillovers from past research [as in Newton: "*If I have seen further it is by standing on the shoulders of Giants*"].

- Such spillovers make the scarce factors embodied in research increasingly productive over time.

Outline

1. **Lab equipment model**
   - Setup
   - Solving the model
   - Planning problem
   - Discussion

2. **Knowledge spillovers model**
   - Setup
   - Solving the model

3. **Growth without scale effects [limited knowledge spillovers]**

Knowledge Spillovers

- Labor L must be allocated between research and production of goods

$$L_R(t) + L_Y(t) = L$$

- New ideas (blueprints) produced by researchers, with spillovers from stock of existing ideas

$$\dot{N}(t) = \eta\, N(t)^{\phi}\, L_R(t), \qquad \phi = 1$$

- One unit of labor in research produces blueprints with arrival rate $\eta N(t)^{\phi}$.

- For now, set the spillover parameter $\phi = 1$.

- Builds in the linearity that gives endogenous growth. Will relax later.

Intermediate Good Producers

- As before, price of intermediate is a markup over marginal cost

$$p(v,t) = \frac{1}{1-\beta}\psi \equiv \bar{p}$$

- But now quantity of each intermediate will depend on size of the goods-producing sector

$$x(v,t) = \left((1-\beta)^2 \frac{A}{\psi}\right)^{1/\beta} L_Y(t) = \bar{x}\, L_Y(t)$$

- Hence static profits depend on size of the goods-producing sector

$$\pi(v,t) = (p(v,t) - \psi)x(v,t) = \beta\bar{p}\, x(v,t) = \beta\bar{p}\bar{x}\, L_Y(t) \equiv \bar{\pi}\, L_Y(t)$$

- Intertemporal profits then as before

$$r(t)V(v,t) = \pi(v,t) + \dot{V}(v,t)$$

Free Entry Condition

- One unit of labor in research produces blueprints with arrival rate $\eta N(t)$.

- Successful innovation results in a perpetual patent, market value $V(v,t)$.

- If market value satisfies $\eta N(t)V(v,t) > w(t)$, strictly positive profits can be made by hiring labor to work on the production of new blueprints.

- With positive entry, $L_R(t) > 0$, we will have

$$\eta N(t)V(v,t) = w(t)$$

Final Good Producers

- First order conditions for aggregate $X(t)$ and $L_Y(t)$ are

$$\begin{aligned}(1-\beta)Y(t) &= P(t)X(t)\\ \beta Y(t) &= w(t)L_Y(t)\end{aligned}$$

- As before the price index for the bundle of intermediates is

$$P(t) = \left(\int_0^{N(t)} p(v,t)^{1-\frac{1}{\beta}}\, dv\right)^{\frac{1}{1-\frac{1}{\beta}}} = N(t)^{\frac{1}{1-\frac{1}{\beta}}}\,\bar{p}$$

- But now the aggregate quantity of intermediates is

$$X(t) = \left(\int_0^{N(t)} x(v,t)^{1-\beta}\, dv\right)^{\frac{1}{1-\beta}} = N(t)^{\frac{1}{1-\beta}}\,\bar{x}\,L_Y(t)$$

- So that

$$P(t)X(t) = N(t)\bar{p}\bar{x}L_Y(t)$$

Wages and Free Entry Condition

- From the first order conditions of final good producers we can write wages

$$
\begin{aligned}
w(t) &= \frac{\beta}{1-\beta}\,\frac{P(t)X(t)}{L_Y(t)} = \frac{\beta}{1-\beta}\bar{p}\bar{x}\,N(t) \\
&= \frac{1}{1-\beta}\bar{\pi}\,N(t)
\end{aligned}
$$

  using the relationship between constants $\bar{\pi} = \beta\bar{p}\bar{x}$.

- Plugging this back into the free-entry condition

$$\eta N(t)V(v,t) = \frac{1}{1-\beta}\,\bar{\pi}\,N(t)$$

- So in any equilibrium with entry the market value of a patent is constant

$$V(v,t) = \frac{1}{\eta(1-\beta)}\bar{\pi}$$

Real Interest Rate and Consumption

- Hence, from the flow version of intertemporal profits, real interest rate is

$$r(t) = \frac{\pi(v,t) + \dot{V}(v,t)}{V(v,t)} = \eta(1-\beta)L_Y(t)$$

- Growth in aggregate consumption $C(t) = c(t)L$ is then given by

$$\frac{\dot{C}(t)}{C(t)} = \frac{r(t) - \rho}{\theta} = \frac{\eta(1-\beta)L_Y(t) - \rho}{\theta}$$

- To pin down the level of consumption we need to use the market clearing condition for the numeraire final good

$$C(t) + \psi \int_0^{N(t)} x(v,t)\, dv = Y(t)$$

which implies

$$C(t) = \left(A\bar{x}^{1-\beta} - \psi\bar{x}\right) N(t) L_Y(t) = b\bar{\pi} N(t) L_Y(t)$$

Outline

Solving the Model

- To summarize, with positive entry the equilibrium dynamics are given by a system of three equations in $C(t), N(t)$ and $L_Y(t)$:

  (i) consumption Euler equation

$$\frac{\dot{C}(t)}{C(t)} = \frac{\eta(1-\beta)L_Y(t) - \rho}{\theta}$$

  (ii) law of motion for new varieties

$$\frac{\dot{N}(t)}{N(t)} = \eta(L - L_Y(t))$$

  (iii) market clearing for final good

$$C(t) = b\bar{\pi}N(t)L_Y(t)$$

- Single initial condition $N(0) > 0$ for predetermined variable $N(t)$. Two jump variables, $C(t)$ and $L_Y(t)$.

Equilibrium Dynamics

- Can reduce this to a single differential equation in $L_Y(t)$, namely

$$\frac{\dot{L}_Y(t)}{L_Y(t)} = \frac{\eta(1-\beta)L_Y(t) - \rho}{\theta} - \eta(L - L_Y(t))$$

  which is strictly increasing in $L_Y(t)$.

- The only way for these dynamics to avoid $L_Y(t) > L$ (which is infeasible) is for $L_Y(t)$ to instantly jump to the constant solution

$$L_Y = \frac{\theta\eta L + \rho}{\theta\eta + \eta(1-\beta)}$$

Balanced Growth Path

- But then if L_Y is constant, so too is the real interest rate

$$r = \eta(1-\beta)L_Y$$

- So with positive entry we have a balanced growth path with

$$\frac{\dot{N}(t)}{N(t)} = \frac{\dot{Y}(t)}{Y(t)} = \frac{\dot{C}(t)}{C(t)} = g > 0$$

where

$$g = \eta L_R = \frac{\eta(1-\beta)L_Y - \rho}{\theta}$$

and

$$L_Y = \frac{\theta\eta L + \rho}{\theta\eta + \eta(1-\beta)}$$

- As in lab equipment model, equilibrium is inefficient.

Balanced Growth Path

- Key comparative statics driven by amount of labor used in research

$$L_R = L - L_Y = \frac{\eta(1-\beta)L - \rho}{\theta\eta + \eta(1-\beta)}$$

- Equilibrium growth $g = \eta L_R$ is then
  - increasing in arrival rate of ideas η, both directly and indirectly via L_R
  - decreasing in rate of time preference ρ
  - increasing in size of labor force L

- Assumptions. For this outcome we need parameters:

  (i) $(1-\beta)\eta L > \rho$, so economy is growing $g > 0$ [with $L_R > 0$]

  (ii) $\rho > (1-\theta)(1-\beta)\eta L$, so TVC is satisfied, $r > g$

- Remark. Otherwise no entry, $L_R = 0$, hence $g = 0$, $L_Y = L$, consumption and output constant at levels pinned down by initial $N(0)$.

Outline

Problems with Scale Effects

- Larger L implies higher r and higher g.
- That is, growth increasing in *level* of resources allocated to research.
- Three problems with this:

  (i) empirically, larger countries do not generally grow faster

  (ii) empirically, level of resources allocated to research steadily increasing but growth rates have not increased — if anything they have slowed down

  (iii) with $n > 0$, growth rate g increases as L increases, perpetual acceleration

Limited Knowledge Spillovers

- Consider limited knowledge spillovers [following Jones 1995]

$$\dot{N}(t) = \eta N(t)^{\phi} L_R(t), \qquad 0 < \phi < 1$$

where

$$L_R(t) + L_Y(t) = L(t) = e^{nt}, \qquad n > 0$$

- Focus on balanced growth path with constant r and labor allocations that are constant *shares* of the total labor force.

- Along such a balanced growth path, the market value of a patent is

$$\begin{aligned} V(v,t) = \int_t^{\infty} e^{-(s-t)r}\, \pi(v,s)\, ds &= \int_t^{\infty} e^{-(s-t)r}\, \bar{\pi} L_Y(s)\, ds \\ &= \int_t^{\infty} e^{-(s-t)r}\, \bar{\pi} L_Y(t) e^{n(s-t)}\, ds \\ &= \frac{\bar{\pi}}{r-n}\, L_Y(t) \end{aligned}$$

Balanced Growth with Limited Spillovers

- Free entry requires

$$\eta N(t)^{\phi} V(v,t) = w(t)$$

- Substituting for the wage and market value value of a patent

$$\eta N(t)^{\phi} \underbrace{\frac{\bar{\pi}}{r-n} L_Y(t)}_{=V(v,t)} = \underbrace{\frac{\bar{\pi}}{1-\beta} N(t)}_{=w(t)}$$

- Let g_N denote growth rate of $N(t)$, to be determined.

- Along a balanced growth path, the free-entry condition implies

$$\phi g_N + n = g_N \qquad \Rightarrow \qquad g_N = \frac{n}{1-\phi}$$

Balanced Growth with Limited Spillovers

- Market clearing condition for final good

$$C(t) = \left(A\bar{x}^{1-\beta} - \psi\bar{x}\right)N(t)L_Y(t) = b\bar{\pi}N(t)L_Y(t)$$

- So aggregate consumption (and output) grows at rate $g_C = g_N + n$ while consumption per worker $c(t) = C(t)/L(t)$ grows at rate g_N.

- Real interest rate is then

$$r = \rho + \theta g_N = \rho + \frac{\theta}{1-\phi}n$$

Discussion

- Balanced growth $g > 0$ with $n > 0$ if limited knowledge spillovers.

- This class of models sometimes known as '*semi-endogenous*' growth.

- If no growth in labor force, i.e., if $n = 0$, limited knowledge spillovers have *level effect* on output but do not generate endogenous growth.

- Sustained growth in labor force $L(t)$ allows sustained growth in labor allocated to research $L_R(t)$, enough to offset the limited knowledge spillovers, delivering sustained growth in $N(t)$.

- *Growth Without Scale Effects?* Two senses in which still scale effects:

  (i) faster growth rate n implies faster growth rate g
  (ii) larger L implies larger Y/L

- Neither of these predictions is clearly present in the data.

Next Class

- Directed technical change.

- Endogenous direction and bias of technical change.

- Implications for skill premia.

Homework

- Consider the endogenous growth model with limited knowledge spillovers.

- Check. *Without* restricting attention to a balanced growth path, show that if the free-entry condition holds, the market value of a patent is

$$V(v,t) = \frac{1}{\eta(1-\beta)}\bar{\pi}\, N(t)^{1-\phi}$$

  Given this, explain why the real interest rate must satisfy the condition

$$r(t) = \eta(1-\beta)L_Y(t)N(t)^{\phi-1} + (1-\phi)\frac{\dot{N}(t)}{N(t)}$$

- Check. Use this condition on the real interest rate to show that, on a balanced growth path, $g_N = n/(1-\phi)$.

- Check. What is the growth rate of this economy if $n = 0$ and $\phi = 1$?