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ECON90003
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Macroeconomics
Tutorial #9: Solutions

1. **Income fluctuations with CARA utility**. Consider a single risk averse household that takes as given a constant interest rate $r > 0$ and that seeks to maximize

$$\mathbb{E}\left\{\sum_{t=0}^{\infty}\beta^t\,u(c_t)\right\}, \qquad 0<\beta<1$$

subject to

$$c_t + a_{t+1} = (1+r)a_t + y_t$$

The household's income $y_t > 0$ fluctuates according to the autoregression

$$y_{t+1} = (1-\phi)\bar{y} + \phi y_t + \varepsilon_{t+1}, \qquad \bar{y} > 0, \qquad 0<\phi<1$$

where the innovations ε_{t+1} are IID $N(0,\sigma_\varepsilon^2)$.

(a) Let $v(a,y)$ denote the household's value function. Setup and explain the Bellman equation that determines $v(a,y)$.

Now suppose the utility function has the constant *absolute* risk aversion (CARA) form

$$u(c) = -\frac{\exp(-\alpha\,c)}{\alpha}, \qquad \alpha > 0$$

(b) Show that the value function that solves the Bellman equation is given by

$$v(a,y) = -\frac{\exp(-A\,(a+By+C))}{A}$$

for some coefficients A, B, C. Solve for the coefficients A, B, C in terms of the parameters.

(c) Let $c(a,y)$ denote the optimal consumption policy function. Solve for $c(a,y)$.

Now consider a Huggett-style incomplete markets model with many such households. Suppose the asset a is in zero net supply.

(d) Define a stationary equilibrium for this economy. Give a computational procedure that would allow you to solve for a stationary equilibrium.

Solutions:

(a) Let $F(y' \,|\, y)$ denote the conditional distribution for the household's income implied by the autoregression. Then the Bellman equation for this problem can be written

$$v(a,y) = \max_{c\geq 0}\Big[u(c) + \beta\int v(a',y')\,dF(y'\,|\,y)\Big]$$

subject to

$$c + a' = (1+r)a + y$$

The Bellman equation characterizes the value $v(a,y)$ of having a assets at the beginning of the period when the household's current income is y, and then proceeding optimally.

(b)-(c) The first order condition for this problem can be written

$$u_1(c) = \beta\int v_1(a',y')\,dF(y'\,|\,y)$$

where it is understood that c satisfies the budget constraint. The envelope condition is

$$v_1(a,y) = \beta R\int v_1(a',y')\,dF(y'\,|\,y)$$

where $R = 1+r$. Hence also $v_1(a,y) = u_1(c)R$. For future reference, note that these conditions imply the consumption Euler equation

$$u_1(c) = \beta R\int u_1(c')\,dF(y'\,|\,y)$$

where again it is understood that c and c' satisfy their respective budget constraints.

Now using the specific CARA utility function $u_1(c) = e^{-\alpha c}$ and using the CARA guess for the value function $v_1(a,y) = e^{-A(a+By+C)}$ we can write the envelope condition as

$$v_1(a,y) = e^{-A(a+By+C)} = Re^{-\alpha c}$$

We can solve this for a *candidate* consumption function, namely

$$c = \frac{A}{\alpha}\Big[a + By + C\Big] + \frac{1}{\alpha}\log R \qquad (*)$$

Of course this is not a solution because we don't yet know A, B, C. Our goal now is to determine these three coefficients.

To determine the three coefficients, first observe that because of the CARA specification we can write $u_1(c) = -\alpha u(c)$ and $v_1(a,y) = -Av(a,y)$. Then using the envelope condition to link these together we have

$$-Av(a,y) = v_1(a,y) = u_1(c)R = -\alpha u(c)R$$

hence

$$u(c) = \frac{A}{\alpha R}\,v(a,y)$$

Next, using the Bellman equation and understanding that c is evaluated at the optimum we have

$$v(a,y) = u(c) + \beta \int v(a',y')\, dF(y' \,|\, y)$$

$$= \frac{A}{\alpha R} v(a,y) + \beta \int \Big\{ -\frac{1}{A} e^{-A(a'+By'+C)} \Big\}\, dF(y' \,|\, y)$$

$$= \frac{A}{\alpha R} v(a,y) + \beta \int \Big\{ -\frac{1}{A} e^{-A(Ra+y-c+By'+C)} \Big\}\, dF(y' \,|\, y)$$

Collecting terms and rearranging

$$\Big(1 - \frac{A}{\alpha R}\Big) v(a,y) = -\beta \frac{1}{A} e^{-A(Ra+y-c+C)} \int e^{-ABy'}\, dF(y' \,|\, y)$$

$$= -\beta \frac{1}{A} e^{-A(Ra+y-c+C)} \int e^{-AB[(1-\phi)\bar{y}+\phi y+\varepsilon']}\, dG(\varepsilon')$$

$$= -\beta \frac{1}{A} e^{-A(Ra+y-c+B[(1-\phi)\bar{y}+\phi y]+C)} \int e^{-AB\varepsilon'}\, dG(\varepsilon')$$

where $G(\varepsilon')$ denotes the distribution of the innovations (shocks) to the income process. Substituting in our guess for the value function then gives the condition

$$\Big(\frac{\alpha R - A}{\alpha R}\Big)\Big(-\frac{1}{A} e^{-A(a+By+C)}\Big) = -\beta \frac{1}{A} e^{-A(Ra+y-c+B[(1-\phi)\bar{y}+\phi y]+C)} \int e^{-AB\varepsilon'}\, dG(\varepsilon')$$

Cancelling common terms and then taking logs gives

$$-A(a+By) + \log\Big(\frac{\alpha R - A}{\alpha R}\Big) = -A(Ra+y-c+B[(1-\phi)\bar{y}+\phi y]) + \log\beta + \log\int e^{-AB\varepsilon'}\, dG(\varepsilon')$$

Cancelling more common terms and rearranging gives the consumption function

$$c = ra + (1-B(1-\phi))y + B(1-\phi)\bar{y} + \frac{1}{A}\Big\{ \log\Big(\frac{\alpha R - A}{\alpha R}\Big) - \log\beta - \log\int e^{-AB\varepsilon'}\, dG(\varepsilon')\Big\} \quad (**)$$

For our guess to work we need the consumption functions $(*)$ and $(**)$ to coincide. Matching coefficients gives a system of three equations in the three unknowns

$$\frac{A}{\alpha} = r \tag{1}$$

$$\frac{AB}{\alpha} = 1 - B(1-\phi) \tag{2}$$

$$\frac{A}{\alpha} C + \frac{1}{\alpha}\log R = B(1-\phi)\bar{y} + \frac{1}{A}\Big\{ \log\Big(\frac{\alpha R - A}{\alpha R}\Big) - \log\beta - \log\int e^{-AB\varepsilon'}\, dG(\varepsilon')\Big\} \tag{3}$$

These three equations pin down the coefficients A, B, C in terms of the underlying parameters $\alpha, r, \phi, \bar{y}$ and the distribution of shocks $G(\varepsilon')$. Notice that we can solve these three equations recursively, first solving (1) to get

$$A = \alpha r$$

then using $A = \alpha r$ in (2) to get

$$B = \frac{1}{1-\phi+r}$$

and then using $A = \alpha r$ and $B = \frac{1}{1-\phi+r}$ in (3) gives

$$rC + \frac{1}{\alpha}\log R = \frac{1-\phi}{1-\phi+r}\bar{y} - \frac{1}{\alpha r}\Big\{\log(\beta R) + \log\int e^{-\alpha\frac{r}{1-\phi+r}\varepsilon'}\,dG(\varepsilon')\Big\}$$

so that

$$C = \frac{1}{r}\left(\frac{1-\phi}{1-\phi+r}\bar{y} - \frac{1}{\alpha r}\Big\{\log(\beta R) + \log\int e^{-\alpha\frac{r}{1-\phi+r}\varepsilon'}\,dG(\varepsilon') + r\log R\Big\}\right)$$

Plugging these solutions back into the consumption function (either $(*)$ or $(**)$ will do) gives the consumption function

$$c(a, y) = ra + \frac{r}{1-\phi+r}y + \frac{1-\phi}{1-\phi+r}\bar{y} - S(r)$$

where the constant

$$S(r) = \frac{1}{\alpha r}\Big\{\log(\beta R) + \log\int e^{-\alpha\frac{r}{1-\phi+r}\varepsilon'}\,dG(\varepsilon')\Big\}$$

is a measure of consumer's additional saving over and above the amount of saving they would do in a benchmark linear-quadratic permanent income model (with $\beta R = 1$). The additional saving $S(r)$ consists of (i) a drift effect from $\log(\beta R)$ which is positive if $\beta R > 1$ (so that the consumer is relatively patient compared to the interest rate) but is negative if $\beta R < 1$ (so that the consumer is relatively impatient compared to the interest rate) and (ii) a precautionary saving effect

$$\log\int e^{-\alpha\frac{r}{1-\phi+r}\varepsilon'}\,dG(\varepsilon') = \frac{1}{2}\alpha^2\Big(\frac{r}{1-\phi+r}\Big)^2\sigma_\varepsilon^2$$

(where here we use that ε' is normally distributed with mean zero and variance σ_ε^2 to calculate the integral). This precautionary saving effect is increasing in risk aversion α, increasing in the innovation variance σ_ε^2 and increasing in the persistence of the shocks ϕ.

(d) This is a bit of a trick question. There is no stationary asset distribution in this model. To see this, observe from the consumption function above that each consumer has asset accumulation policy

$$a' = g(a, y) = (1+r)a + y - c(a, y) = a + \frac{1-\phi}{1-\phi+r}(y - \bar{y}) + S(r)$$

or in time series notation

$$a_{t+1} - a_t = \frac{1-\phi}{1-\phi+r}(y_t - \bar{y}) + S(r)$$

In this sense, each individual's assets follow a random walk with increments given by $y_t - \bar{y}$. For example, if $\phi = 0$ so that y_t is IID over time, each individual's assets follow a random walk with increments $\frac{1}{1+r}\varepsilon_t$. We cannot expect this process to yield a stationary asset distribution in the usual sense. But since assets are in zero net supply, when we sum over all households we get

$$0 = \int \frac{1-\phi}{1-\phi+r}(y-\bar{y})\, d\bar{F}(y) + S(r)$$

where $\bar{F}(y)$ denotes the stationary income distribution implied by $F(y' \,|\, y)$. But this distribution has mean $\int y\, d\bar{F}(y) = \bar{y}$ so that we conclude that

$$0 = S(r)$$

In other words, the equilibrium interest rate $r > 0$ that clears the asset market is such that the 'additional saving' term $S(r) = 0$, or put differently, the drift effect and the precautionary saving effect must exactly offset so that

$$0 = \log(\beta(1+r)) + \frac{1}{2}\alpha^2\Big(\frac{r}{1-\phi+r}\Big)^2\sigma_\varepsilon^2$$

Writing this as

$$-\log(\beta(1+r)) = \frac{1}{2}\alpha^2\Big(\frac{r}{1-\phi+r}\Big)^2\sigma_\varepsilon^2 > 0$$

we see that since the precautionary saving effect is positive, we must have negative drift $\beta(1+r) < 1$ so that the equilibrium interest rate is low relative to the rate of time preference. Using $-\log\beta \approx \rho > 0$ and $-\log(1+r) \approx -r < 0$ we can write this condition as

$$\rho - r \approx \frac{1}{2}\alpha^2\Big(\frac{r}{1-\phi+r}\Big)^2\sigma_\varepsilon^2$$

The LHS is strictly decreasing from 0 at $r = \rho$ to $-\infty$ as $r \to \infty$ while the RHS is strictly increasing from 0 at $r = 0$ to $\frac{1}{2}\alpha^2\sigma_\varepsilon^2 > 0$ as $r \to \infty$. Hence there is a unique $r \in (0, \rho)$ such that $S(r) = 0$ and the asset market clears. In short, for this CARA example, we are able to determine the equilibrium interest rate without being able to construct a stationary asset distribution. Note that the LHS is strictly increasing in risk aversion α, the innovation variance σ_ε^2, and in the persistence of the shocks ϕ. Higher values of any of these parameters will strengthen the precautionary saving effect and reduce the equilibrium r.

To relate this back to the usual permanent income model, observe that in equilibrium with r such that $S(r) = 0$ individual consumption simplifies to

$$c_t = ra_t + \frac{r}{1-\phi+r}y_t + \frac{1-\phi}{1-\phi+r}\bar{y}$$

with asset accumulation

$$\Delta a_{t+1} = a_{t+1} - a_t = \frac{1-\phi}{1-\phi+r}(y_t - \bar{y})$$

Differencing individual consumption gives

$$\Delta c_{t+1} = r\Delta a_{t+1} + \frac{r}{1-\phi+r}\Delta y_{t+1}$$

Substituting in the asset accumulation

$$\Delta c_{t+1} = r\frac{1-\phi}{1-\phi+r}(y_t - \bar{y}) + \frac{r}{1-\phi+r}\Delta y_{t+1}$$

But from the autoregression for income

$$\Delta y_{t+1} = -(1-\phi)(y_t - \bar{y}) + \varepsilon_{t+1}$$

Hence we get

$$\Delta c_{t+1} = r\frac{1-\phi}{1-\phi+r}(y_t - \bar{y}) + \frac{r}{1-\phi+r}\Delta y_{t+1}$$

$$= r\frac{1-\phi}{1-\phi+r}(y_t - \bar{y}) + \frac{r}{1-\phi+r}\left(-(1-\phi)(y_t - \bar{y}) + \varepsilon_{t+1}\right)$$

$$= \frac{r}{1-\phi+r}\varepsilon_{t+1}$$

or

$$c_{t+1} = c_t + \frac{r}{1-\phi+r}\varepsilon_{t+1}$$

To summarise, *in equilibrium* each consumer *behaves as if* they follow a strict version of the linear-quadratic permanent income model where consumption follows a random walk. This is because their positive precautionary saving incentives are exactly offset by the negative drift from $\beta(1+r) < 1$.

To see this another way, recall the consumption Euler equation

$$u_1(c_t) = \beta R \mathbb{E}_t\{u_1(c_{t+1})\}$$

so that with CARA utility

$$e^{-\alpha c_t} = \beta R \mathbb{E}_t\{e^{-\alpha c_{t+1}}\}$$

But since consumption follows the random walk given above

$$e^{-\alpha c_t} = \beta R \mathbb{E}_t\{e^{-\alpha(c_t + \frac{r}{1-\phi+r}\varepsilon_{t+1})}\}$$

or

$$e^{-\alpha c_t} = \beta R e^{-\alpha c_t} \mathbb{E}_t\{e^{-\alpha \frac{r}{1-\phi+r}\varepsilon_{t+1}}\}$$

or

$$1 = \beta R \mathbb{E}_t\{e^{-\alpha \frac{r}{1-\phi+r}\varepsilon_{t+1}}\}$$

Calculating the expectation then gives the condition

$$1 = \beta R e^{\frac{1}{2}\alpha^2(\frac{r}{1-\phi+r})^2\sigma_\varepsilon^2}$$

But this is equivalent to the condition that $S(r) = 0$ so that the drift term and precautionary savings term exactly offset.