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ECON90003
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Macroeconomics
Tutorial #10: Solutions

1. **Tauchen-Hussey by hand.** Consider the continuous-state AR(1) process

$$x_{t+1} = (1-\rho)\bar{x} + \rho x_t + \varepsilon_{t+1}, \qquad 0 < \rho < 1$$

where the innovations ε_t are

$$\varepsilon_t \sim \text{IID } N(0, \sigma^2)$$

This process is completely characterized by three parameters: $\bar{x}, \rho$ and σ. Now consider a 2-state Markov chain on x_i, $i = 1, 2$ with transition probabilities $p_{ij} = \text{Prob}[x_{t+1} = x_j \,|\, x_t = x_i]$ for $i, j = 1, 2$ given by a symmetric matrix of the form

$$\begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} = \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}, \qquad 0 < p < 1$$

This 2-state Markov chain is completely characterized by three parameters, x_1, x_2 and p. We can make this 2-state Markov chain *mimic* the continuous-state AR(1) by choosing the three parameters x_1, x_2 and p to match three moments of the AR(1). The natural moments to target are the unconditional mean $\mathbb{E}[x]$, unconditional variance $\text{Var}[x]$, and AR(1) coefficient.

Using this approach, derive expressions for the three parameters of the Markov chain x_1, x_2 and p that will allow you to mimic a continuous-state AR(1) with parameters $\bar{x}, \rho$ and σ.

Solution:

Let's first compute the stationary distribution for this Markov chain. A stationary distribution $\bar{\boldsymbol{\pi}}$ satisfies

$$\bar{\boldsymbol{\pi}} = \boldsymbol{P}^\top \bar{\boldsymbol{\pi}}$$

or

$$(\boldsymbol{I} - \boldsymbol{P}^\top)\bar{\boldsymbol{\pi}} = \boldsymbol{0}$$

So for this symmetric 2-by-2 case

$$\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}\right]\begin{pmatrix} \bar{\pi}_1 \\ \bar{\pi}_2 \end{pmatrix} = \begin{pmatrix} 1-p & p-1 \\ p-1 & 1-p \end{pmatrix}\begin{pmatrix} \bar{\pi}_1 \\ \bar{\pi}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Which reduces to

$$\begin{aligned} (1-p)\bar{\pi}_1 - (1-p)\bar{\pi}_2 &= 0 \\ -(1-p)\bar{\pi}_1 + (1-p)\bar{\pi}_2 &= 0 \end{aligned}$$

The first of these tells us that $\bar{\pi}_1 = \bar{\pi}_2$. The second of these is a scalar multiple of the first and so has no extra information. But these probabilities must sum to one, $\bar{\pi}_1 + \bar{\pi}_2 = 1$, so we have

$\bar{\pi}_1 = \bar{\pi}_2 = 0.5$. Averaged over a long horizon this symmetric 2-state Markov chain spends half the time in each state.

Using this stationary distribution we can compute the unconditional moments of the Markov chain and choose parameters to align these moments with their counterparts from the continuous-state AR(1). In particular, the unconditional average of the Markov chain is

$$\mathbb{E}[x] = \frac{1}{2}(x_1 + x_2)$$

and we should set this to be equal to the unconditional average of the AR(1), namely $\bar{x}$. Using symmetry, set $x_1 = \bar{x} - h$ and $x_2 = \bar{x} + h$ for some h to be determined. Likewise the unconditional variance of the Markov chain is

$$\begin{aligned}\text{Var}[x] &= \mathbb{E}[x^2] - \mathbb{E}[x]^2 \\ &= \frac{1}{2}(x_1^2 + x_2^2) - \left(\frac{1}{2}(x_1 + x_2)\right)^2 \\ &= \frac{1}{2}((\bar{x} - h)^2 + (\bar{x} + h)^2) - \bar{x}^2 \\ &= h^2\end{aligned}$$

Hence we should choose h to equal the unconditional standard deviation of the AR(1), namely $\sigma/\sqrt{1-\rho^2}$. In short, the first two moments of the AR(1) pin down the support of the Markov chain

$$x_1 = \bar{x} - \frac{\sigma}{\sqrt{1-\rho^2}}, \qquad \text{and} \qquad x_2 = \bar{x} + \frac{\sigma}{\sqrt{1-\rho^2}}$$

We have one remaining parameter to determine, the transition probability p. Our goal is to choose p so that the Markov chain has the same serial correlation properties as the continuous state AR(1). To that end, note that the AR(1) coefficient ρ is

$$\rho = \frac{\text{Cov}[x_t, x_{t+1}]}{\text{Var}[x_t]} = \frac{\mathbb{E}[x_t x_{t+1}] - \mathbb{E}[x_t]^2}{\text{Var}[x_t]}$$

We already know to set $\mathbb{E}[x_t] = \bar{x}$ and $\text{Var}[x_t] = h^2$ so our main task it to calculate $\mathbb{E}[x_t x_{t+1}]$. By the law of iterated expectations this is

$$\mathbb{E}[x_t x_{t+1}] = \mathbb{E}\{x_t \mathbb{E}[x_{t+1} \,|\, x_t]\}$$

where the outer expectation is taken with respect to the unconditional distribution of x_t. For the Markov chain, the conditional expectations are

$$\mathbb{E}[x_{t+1} \,|\, x_t = x_1] = p x_1 + (1-p) x_2$$

and

$$\mathbb{E}[x_{t+1} \,|\, x_t = x_2] = (1-p) x_1 + p x_2$$

Hence the unconditional expectation we seek is

$$\begin{aligned}
\mathbb{E}[x_t x_{t+1}] &= \frac{1}{2}x_1(px_1 + (1-p)x_2) + \frac{1}{2}x_2((1-p)x_1 + px_2) \\
&= \frac{1}{2}px_1x_1 + (1-p)x_1x_2 + \frac{1}{2}px_2x_2 \\
&= \frac{1}{2}p(\bar{x}^2 - 2\bar{x}h + h^2) + (1-p)(\bar{x}-h)(\bar{x}+h) + \frac{1}{2}p(\bar{x}^2 + 2\bar{x}h + h^2) \\
&= p(\bar{x}^2 + h^2) + (1-p)(\bar{x}^2 - h^2) \\
&= \bar{x}^2 + (2p-1)h^2
\end{aligned}$$

So the implied AR(1) coefficient is

$$\begin{aligned}
\frac{\mathbb{E}[x_t x_{t+1}] - \mathbb{E}[x_t]^2}{\text{Var}[x_t]} &= \frac{\bar{x}^2 + (2p-1)h^2 - \bar{x}^2}{h^2} \\
&= 2p - 1
\end{aligned}$$

So we should choose p such that $2p - 1 = \rho$, that is, we should choose

$$p = \frac{1+\rho}{2}$$

(e.g., if $\rho = 0$, set $p = 0.5$; if $\rho = 0.9$ set $p = 0.85$, etc).

2. **Discount factor shocks.** Consider a Huggett-style incomplete markets model where individual households face *discount factor* shocks. In particular, their time discount factor $\beta_t \in (0,1)$ evolves according to a Markov chain with transition probabilities $\phi(\beta' \,|\, \beta) = \text{Prob}[\beta_{t+1} = \beta' \,|\, \beta_t = \beta]$. Their preferences are represented recursively by

$$v_t = u(c_t) + \beta_t \mathbb{E}_t\big[v_{t+1}\big]$$

which they maximize subject to the sequence of budget constraints

$$c_t + qa_{t+1} \leq a_t + y_t$$

and a borrowing constraint of the form $a_{t+1} \geq -\phi$ for some ϕ. Their income $y_t > 0$ evolves exogenously according to a Markov chain with transition probabilities $\pi(y' \,|\, y) = \text{Prob}[y_{t+1} = y' \,|\, y_t = y]$.

(a) Let $v(a, y, \beta)$ denote the value function of a household of type a, y when their discount factor is β. Setup the household dynamic programming problem in terms of this value function and define a stationary equilibrium for this economy.

(b) Give a computational procedure that would allow you to solve for a stationary equilibrium.

Solutions:

(a) The Bellman equation for an individual of type a, y, β given q can be written

$$v(a, y, \beta\,;\, q) = \max_{a' \geq -\phi} \Big[u(c) + \beta \sum_{y'} \sum_{\beta'} v(a', y', \beta'\,;\, q)\, \pi(y' \,|\, y) \phi(\beta' \,|\, \beta) \Big]$$

subject to

$$c + qa' \leq a + y$$

Let $a' = g(a, y, \beta)$ denote the policy function implied by the maximization on the RHS of the Bellman equation. A *stationary equilibrium* is a value function $v(a, y, \beta)$, policy function $g(a, y, \beta)$, distribution $\mu(a, y, \beta)$ and price q such that:

(i) taking q as given, $v(a, y, \beta)$ and $g(a, y, \beta)$ solve the dynamic programming problem for an individual of type a, y, β

(ii) the asset market clears

$$\sum_a \sum_y \sum_\beta g(a, y, \beta) \mu(a, y, \beta) = 0$$

(iii) the distribution $\mu(a, y, \beta)$ is stationary

$$\mu(a', y', \beta') = \sum_a \sum_y \sum_\beta \text{Prob}[a', y', \beta' \,|\, a, y, \beta]\, \mu(a, y, \beta)$$

where the conditional distribution $\text{Prob}[a', y', \beta' \,|\, a, y, \beta]$ is given by the policy function $a' = g(a, y, \beta)$ and the exogenous distributions $\pi(y' \,|\, y)$ and $\phi(\beta' \,|\, \beta)$.

(b) Start with an initial guess q^0. Solve the individual dynamic programming problem for $v(a, y, \beta\,;\, q^0)$ and $g(a, y, \beta\,;\, q^0)$ given q^0. Using the solution to the individual dynamic programming problem, solve for the stationary distribution $\mu(a, y, \beta\,;\, y^0)$ implied by $g(a, y, \beta\,;\, y^0)$ and the exogenous $\pi(y' \,|\, y)$ and $\phi(\beta' \,|\, \beta)$. Then compute the error on the market-clearing condition

$$\Big\| \sum_a \sum_y \sum_\beta g(a, y, \beta\,;\, q^0) \mu(a, y, \beta\,;\, q^0) \Big\|$$

If this error is less than some pre-specified *tolerance*, stop. Otherwise update to q^1 and try again. The updating should be as follows: if for any q^j (for $j = 0, 1, 2, \dots$) we have

$$\sum_a \sum_y \sum_\beta g(a, y, \beta\,;\, q^j) \mu(a, y, \beta\,;\, q^j) > 0$$

then there is *excess savings* and so we should update to some $q^{j+1} > q^j$. Likewise if for any q^j we have

$$\sum_a \sum_y \sum_\beta g(a, y, \beta\,;\, q^j) \mu(a, y, \beta\,;\, q^j) < 0$$

then there is *excess borrowing* and so we should update to some $q^{j+1} < q^j$.