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ECON90003
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Macroeconomics
Tutorial #7: Solutions

1. **Complete markets with CRRA preferences.** Suppose there are $i = 1, ..., I$ individuals with stochastic endowments $y_t^i(s^t)$ given by probabilities $\pi_t(s^t)$ and that these individuals all evaluate payoffs using the same expected utility function

$$U(c^i) = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u(c_t^i(s^t))\pi_t(s^t), \qquad 0 < \beta < 1$$

Moreover suppose that $u(c)$ has the constant coefficient of relative risk aversion (CRRA) form

$$u(c) = \frac{c^{1-\sigma} - 1}{1-\sigma}, \qquad \sigma > 0$$

(a) Consider a social planner that chooses $c^i = \{c_t^i(s^t)\}_{t=0}^{\infty}$ for each i to maximize the social welfare function

$$W = \sum_i \lambda_i U(c^i)$$

subject to the sequence of resource constraints

$$\sum_i c_t^i(s^t) \leq \sum_i y_t^i(s^t)$$

where the $\lambda_i \geq 0$ denote a set of given welfare weights. Solve for the planner's consumption allocation. What are the key *cross-sectional* properties of the consumption allocation? What are the key *time-series* properties of the consumption allocation? Explain how these depend on the coefficient of risk aversion σ and on the properties of the endowment processes $y_t^i(s^t)$.

(b) Now consider an Arrow-Debreu market economy where individuals can trade at time $t = 0$ in a complete set of contingent claims with prices $q_t^0(s^t)$ subject to the single intertemporal budget constraint

$$\sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) c_t^i(s^t) \leq \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) y_t^i(s^t)$$

Let $\mu_i \geq 0$ denote the multiplier on an individual's intertemporal budget constraint. Solve for the equilibrium consumption allocation and the equilibrium prices. Explain how these compare to their counterparts in the planner's problem. Solve for the equilibrium multipliers μ_i. Explain how these depend on the coefficient of risk aversion σ and on the properties of the endowment processes $y_t^i(s^t)$.

SOLUTIONS:

(a) Let $\theta_t(s^t) \geq 0$ denote the multipliers on the resource constraints facing the planner so that we can write the Lagrangian

$$\mathcal{L} = \sum_{t=0}^{\infty} \sum_{s^t} \sum_{i} \left\{ \lambda_i \beta^t u(c_t^i(s^t))\pi_t(s^t) + \theta_t(s^t)(y_t^i(s^t) - c_t^i(s^t)) \right\}$$

The key first order condition for this problem can be written

$$\lambda_i \beta^t u'(c_t^i(s^t))\pi_t(s^t) = \theta_t(s^t)$$

Hence for individual i and individual $i = 1$

$$\frac{\lambda_i}{\lambda_1} \frac{u'(c_t^i(s^t))}{u'(c_t^1(s^t))} = 1$$

In the specific case of CRRA preferences with $u'(c) = c^{-\sigma}$ this implies

$$\frac{\lambda_i}{\lambda_1} \left(\frac{c_t^i(s^t)}{c_t^1(s^t)} \right)^{-\sigma} = 1$$

or

$$c_t^i(s^t) = \left(\frac{\lambda_i}{\lambda_1} \right)^{1/\sigma} c_t^1(s^t)$$

Summing over i and using the resource constraints then gives

$$\sum_i c_t^i(s^t) = \sum_i \left(\frac{\lambda_i}{\lambda_1} \right)^{1/\sigma} c_t^1(s^t) = \sum_i y_t^i(s^t) = Y_t(s^t)$$

which allows us to pin down $c_t^1(s^t)$. In particular

$$c_t^1(s^t) = Y_t(s^t) \bigg/ \sum_i \left(\frac{\lambda_i}{\lambda_1} \right)^{1/\sigma}$$

so that we have the allocations

$$c_t^i(s^t) = \left(\frac{\lambda_i^{1/\sigma}}{\sum_i \lambda_i^{1/\sigma}} \right) Y_t(s^t)$$

Each individual gets a fixed, time-invariant, share of the aggregate endowment with the size of that share increasing in their welfare weight λ_i. The cross-sectional distribution of consumption does not change over time and the constant amount of dispersion in the cross-sectional distribution of consumption is given by the underlying dispersion in λ_i and the amount of risk aversion σ. In particular, the standard deviation of log consumption is $1/\sigma$ times the standard deviation of log λ_i so that when risk aversion is high there is less dispersion in the cross-sectional distribution of consumption and when risk aversion is low there is more dispersion in the cross-sectional distribution of consumption. In this allocation, the amount of idiosyncratic risk that individuals are exposed to is completely eliminated leaving them only exposed to aggregate risk. Consequently, all of the time-series properties of individual consumption come from the aggregate endowment (e.g., any serial correlation in individual consumption comes from serial correlation in $Y_t(s^t)$ etc).

(b) In the Arrow-Debreu setting an individual's Lagrangian can be written

$$\mathcal{L} = \sum_{t=0}^{\infty} \sum_{s^t} \left\{ \beta^t u(c_t^i(s^t))\pi_t(s^t) + \mu_i q_t^0(s^t)(y_t^i(s^t) - c_t^i(s^t)) \right\}$$

The key first order condition for this problem can be written

$$\beta^t u'(c_t^i(s^t))\pi_t(s^t) = \mu_i q_t^0(s^t)$$

And following the same arguments as in part (a) with $u'(c) = c^{-\sigma}$ we get

$$c_t^i(s^t) = \left(\frac{\mu_i^{-1/\sigma}}{\sum_i \mu_i^{-1/\sigma}} \right) Y_t(s^t)$$

The same as in part (a) but with $\lambda_i = 1/\mu_i$. Notice that at this allocation indeed we have

$$\frac{c_t^i(s^t)^{-\sigma}}{\mu_i} = \left(\frac{1}{\sum_i \mu_i^{-1/\sigma}} \right)^{-\sigma} Y_t(s^t)^{-\sigma}, \qquad \text{for all } i$$

So equilibrium prices are

$$q_t^0(s^t) = \beta^t \pi_t(s^t) \left(\frac{1}{\sum_i \mu_i^{-1/\sigma}} \right)^{-\sigma} Y_t(s^t)^{-\sigma}$$

(i.e., there is a part β^t reflecting pure time discounting, a part $\pi_t(s^t)$ reflecting the probabilities of each s^t and a part reflecting the marginal utility of the aggregate endowment in that state, $u'(Y) = Y^{-\sigma}$). Now let

$$\omega_i \equiv \left(\frac{\mu_i^{-1/\sigma}}{\sum_i \mu_i^{-1/\sigma}} \right)$$

denote the fixed share that each individual gets of the aggregate endowment so that $c_t^i(s^t) = \omega_i Y_t(s^t)$. These constants are inversely proportional to the Lagrange multipliers and still need to be determined. Plugging $c_t^i(s^t) = \omega_i Y_t(s^t)$ into the intertemporal budget constraints gives

$$\sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t)\, \omega_i Y_t(s^t) = \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) y_t^i(s^t)$$

Hence

$$\omega_i = \frac{\sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) y_t^i(s^t)}{\sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) Y_t(s^t)}$$

Thus each individual's consumption share ω_i is given by their share of the economy's total intertemporal wealth. Then plugging in the solution for equilibrium prices above we have

$$\omega_i = \frac{\sum_{t=0}^{\infty} \sum_{s^t} \beta^t y_t^i(s^t) Y_t(s^t)^{-\sigma} \pi_t(s^t)}{\sum_{t=0}^{\infty} \sum_{s^t} \beta^t Y_t(s^t)^{1-\sigma} \pi_t(s^t)}$$

The key point is that individual's with a high share ω_i of the economy's intertemporal wealth will have a high consumption share so that in this sense the market economy acts 'like' a planner who gives high weight λ_i to those who have a high share of the economy's intertemporal wealth and gives low weight to those who do not.

2. **Existence of a representative consumer.** Consider a static economy with two individuals $i = 1, 2$ with utility functions $u_i(c_i)$ that are strictly increasing and strictly concave in consumption c_i. Consider the simple planning problem

$$W(y) = \max_{c_1, c_2} \left[\lambda_1 u_1(c_1) + \lambda_2 u_2(c_2) \right]$$

subject to the resource constraint

$$c_1 + c_2 \leq y$$

(a) Show that the solution of this problem is a strictly increasing strictly concave function $W(y)$ which depends on the weights λ_i. Derive a formula for $W'(y)$.

(b) Suppose that both individuals have utility functions that belong to the class of constant *absolute* risk aversion (CARA) utility functions

$$u_i(c_i) = -\frac{\exp(-\alpha_i c_i)}{\alpha_i}, \qquad \alpha_i > 0$$

with coefficients of absolute risk aversion α_i that potentially differs across individuals. Solve for $W(y)$. Explain how this function depends on the weights λ_i and the risk aversion coefficients α_i. Let $c = c_1 + c_2$ denote aggregate consumption and let $U(c)$ denote the utility of the '*representative consumer*' constructed in this way. In what sense is $U(c)$ representative? Does $U(c)$ belong to the class of CARA utility functions?

Solutions:

(a) Since the objective is strictly concave and the constraint set is convex by the maximum theorem the function $W(y)$ is strictly concave and the set of maximizers is single-valued. To characterize the solution a bit further, let

$$W(y) = \max_{0 \leq x \leq y} \left[\lambda_1 u_1(x) + \lambda_2 u_2(y - x) \right]$$

denote the maximum value and let

$$c(y) = \operatorname*{argmax}_{0 \leq x \leq y} \left[\lambda_1 u_1(x) + \lambda_2 u_2(y - x) \right]$$

denote the choice of c_1 that achieves the maximum, i.e., $c_1 = c(y)$ and $c_2 = y - c(y)$. The first order condition for this problem can be written

$$\lambda_1 u_1'(c_1) = \lambda_2 u_2'(c_2), \qquad c_1 + c_2 = y$$

This implicitly determines $c_1 = c(y)$ via

$$\lambda_1 u_1'(c(y)) = \lambda_2 u_2'(y - c(y))$$

Notice that from the implicit function theorem

$$c'(y) = \frac{\lambda_2 u_2''(c_2)}{\lambda_1 u_1''(c_1) + \lambda_2 u_2''(c_2)} \in (0, 1)$$

so that an increase in y increases both c_1 and c_2. By the envelope theorem

$$W'(y) = \lambda_2 u_2'(y - c(y)) > 0$$

(b) With the CARA specification, marginal utility is $u_i'(c) = \exp(-\alpha_i c)$ so the key first order condition for the planner becomes

$$\lambda_1 \exp(-\alpha_1 c_1) = \lambda_2 \exp(-\alpha_2 c_2)$$

so that on taking logs and using $c_1 + c_2 = y$

$$\log \lambda_1 - \alpha_1 c_1 = \log \lambda_2 - \alpha_2 (y - c_1)$$

which solves for

$$c_1 = \frac{1}{\alpha_1 + \alpha_2} \left[\log \left(\frac{\lambda_1}{\lambda_2} \right) + \alpha_2 y \right]$$

hence

$$c_2 = \frac{1}{\alpha_1 + \alpha_2} \left[\log \left(\frac{\lambda_2}{\lambda_1} \right) + \alpha_1 y \right]$$

Plugging these solutions into the planner's objective then gives

$$W(y) = -\frac{\lambda_1}{\alpha_1} \exp\left(-\frac{\alpha_1}{\alpha_1 + \alpha_2} \left[\log \left(\frac{\lambda_1}{\lambda_2} \right) + \alpha_2 y \right] \right) - \frac{\lambda_2}{\alpha_2} \exp\left(-\frac{\alpha_2}{\alpha_1 + \alpha_2} \left[\log \left(\frac{\lambda_2}{\lambda_1} \right) + \alpha_1 y \right] \right)$$

which after some tedious algebra simplifies to

$$W(y) = -\frac{\alpha_1 + \alpha_2}{\alpha_1 \alpha_2} \exp\left(-\frac{\alpha_1 \alpha_2}{\alpha_1 + \alpha_2}\, y \right) \left[\lambda_1^{\frac{\alpha_2}{\alpha_1 + \alpha_2}} \lambda_2^{\frac{\alpha_1}{\alpha_1 + \alpha_2}} \right]$$

So with $c = y$ we can say that the utility function $U(c)$ of the representative consumer is

$$U(c) = -\frac{\alpha_1 + \alpha_2}{\alpha_1 \alpha_2} \exp\left(-\frac{\alpha_1 \alpha_2}{\alpha_1 + \alpha_2}\, c \right) \left[\lambda_1^{\frac{\alpha_2}{\alpha_1 + \alpha_2}} \lambda_2^{\frac{\alpha_1}{\alpha_1 + \alpha_2}} \right]$$

The utility function $U(c)$ is of the CARA class with composite risk aversion

$$\alpha^* = \frac{\alpha_1 \alpha_2}{\alpha_1 + \alpha_2}$$

Here the representative consumer is less risk averse than *either* of the underlying preferences, $\alpha^* < \min[\, \alpha_1 \,,\, \alpha_2 \,]$. In this sense, the risk aversion of the representative consumer may not accurately reflect the risk aversion in the underlying preferences.