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ECON90003
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Macroeconomics
Tutorial #6: Solutions

1. **McCall model with wage growth.** Consider an unemployed worker with risk neutral preferences

$$\mathbb{E}\left\{\sum_{t=0}^{\infty}\beta^t\, c_t\right\}, \qquad 0<\beta<1$$

Each period the worker draws an IID initial wage offer w from a distribution $F(w)$. If they accept the wage offer they become employed and have wage $w_t = wg^t$ growing at $g \geq 1$ thereafter. If they reject the wage offer they remain unemployed and draw a new initial wage offer w' next period. Assume $\beta g < 1$.

(a) Let $v(w)$ denote the unemployed worker's value function. Setup the unemployed worker's dynamic programming problem in terms of this value function.

(b) Show that the unemployed worker's problem is characterized by a reservation wage $\bar{w}$ such that the worker rejects the offer if $w < \bar{w}$ and accepts the offer if $w > \bar{w}$. Provide an expression for $v(w)$ in terms of $\bar{w}$.

(c) Consider two economies exactly the same except for different rates of wage growth, $g_1 > g_2$ say. Which economy has the higher reservation wage? Explain.

SOLUTIONS:

(a) Accepting an offer of w that grows at g implies consumption $c_t = wg^t$ with payoff

$$\sum_{t=0}^{\infty}\beta^t w g^t = \frac{1}{1-\beta g}\, w$$

Hence the Bellman equation for an unemployed worker is

$$v(w) = \max_{\text{accept, reject}}\left[\frac{1}{1-\beta g}w\,,\; b+\beta\int_0^{\infty} v(w')\,dF(w')\right]$$

(b) Rejecting a wage offer gives an expected payoff

$$b+\beta\int_0^{\infty} v(w')\,dF(w')$$

that is independent of the current w. Hence it is optimal to accept all w such that

$$\frac{1}{1-\beta g}w > b+\beta\int_0^{\infty} v(w')\,dF(w')$$

and to reject all w such that

$$\frac{1}{1-\beta g} w < b + \beta \int_0^\infty v(w')\, dF(w')$$

In other words, there is a constant $\bar{w}$ such that it is optimal to accept all $w > \bar{w}$ and to reject all $w < \bar{w}$ where $\bar{w}$ satisfies the indifference condition

$$\frac{1}{1-\beta g} \bar{w} = b + \beta \int_0^\infty v(w')\, dF(w')$$

Hence in terms of $\bar{w}$ the value function $v(w)$ has the piecewise linear form

$$v(w) = \begin{cases} \dfrac{1}{1-\beta g}\bar{w} & w \leq \bar{w} \\ \\ \dfrac{1}{1-\beta g} w & w \geq \bar{w} \end{cases}$$

(c) To answer this question we need to determine the comparative statics of $\bar{w}$ with respect to g. To get a nice expression characterizing $\bar{w}$ we first write the indifference condition as

$$\frac{1}{1-\beta g}\bar{w} = b + \frac{\beta}{1-\beta g}\left(\int_0^{\bar{w}} \bar{w}\, dF(w') + \int_{\bar{w}}^\infty w'\, dF(w')\right)$$

or

$$\frac{1-\beta}{1-\beta g}\bar{w}\int_0^{\bar{w}} dF(w') - b = \frac{\beta}{1-\beta g}\int_{\bar{w}}^\infty w'\, dF(w') - \frac{1}{1-\beta g}\bar{w}\int_{\bar{w}}^\infty dF(w')$$

so that

$$\frac{1-\beta}{1-\beta g}\bar{w} - b = \frac{\beta}{1-\beta g}\int_{\bar{w}}^\infty w'\, dF(w') - \frac{1}{1-\beta g}\bar{w}\int_{\bar{w}}^\infty dF(w') + \frac{1-\beta}{1-\beta g}\bar{w}\int_{\bar{w}}^\infty dF(w')$$

Collecting terms on the RHS this is

$$\frac{1-\beta}{1-\beta g}\bar{w} - b = \frac{\beta}{1-\beta g}\int_{\bar{w}}^\infty (w' - \bar{w})\, dF(w')$$

or

$$\bar{w} - \left(\frac{1-\beta g}{1-\beta}\right) b = \frac{\beta}{1-\beta}\int_{\bar{w}}^\infty (w' - \bar{w})\, dF(w')$$

The RHS (expected benefit of searching again) is a strictly decreasing function of $\bar{w}$ from $\frac{\beta}{1-\beta}\mathbb{E}\{w\}$ at $\bar{w} = 0$ to 0 at $\bar{w} = +\infty$. The LHS (opportunity cost of accepting $w = \bar{w}$) is a linearly increasing function of $\bar{w}$. Hence by the intermediate value theorem there is a unique reservation wage $\bar{w}$ that solves this equation and that

$$\bar{w} \geq \left(\frac{1-\beta g}{1-\beta}\right) b$$

The RHS of the indifference condition is independent of g. A higher g shifts the LHS up, hence the $\bar{w}$ that solves the indifference condition is lower. That is, $\bar{w}$ is decreasing in g. The intuition for this is that the sooner a job is accepted, the sooner the worker benefits from the wage growth. We conclude that the economy with $g_1 > g_2$ will have $\bar{w}_1 < \bar{w}_2$.

2. **Diamond-Mortensen-Pissarides model.** Consider a search model of the labor market in discrete time $t = 0, 1, 2....$ Risk neutral workers and firms have common discount factor $\beta \in (0,1)$. Workers and firms are matched via a standard constant-returns-to-scale matching function $M(u,v)$ where u_t denotes the unemployment rate and v_t the vacancy rate at time t. When a match forms, a firm is able to produce a constant amount of output $z > 0$. The worker receives a wage of w_t and the firm makes a flow profit of $z - w_t$. Job matches between workers and firms are destroyed at an exogenous rate $\delta \in (0,1)$. Firms can create jobs by posting vacancies with a per period cost κz proportional to z. There is free-entry into job creation. When unemployed, workers receive constant flow utility $b \leq w_t$ from unemployment benefits.

   (a) Let V_t, J_t denote the time t value to a firm of a vacancy and a filled job respectively and let U_t, W_t denote the value to a worker of unemployment and employment respectively. Setup and explain the Bellman equations that determine the evolution of these four values over time.

   Now suppose that wages are determined by Nash-Bargaining between a worker and firm such that in equilibrium the worker's surplus is a constant fraction $\phi \in (0,1)$ of the total match surplus

$$W_t - U_t = \phi S_t, \qquad S_t \equiv W_t - U_t + J_t - V_t$$

   Suppose also that free entry drives the value of a vacancy to $V_t = 0$ for all t.

   (b) Let the matching function be $M(u,v) = u^{\alpha} v^{1-\alpha}$. Explain how the steady state wage, labor market tightness $\theta = v/u$ and unemployment rate are determined.

   (c) Now suppose productivity increases from z to $z' > z$. Explain what happens to the steady-state wage, labor market tightness, unemployment rate, vacancy rate, and vacancy filling rate. What about if β decreases to $\beta' < \beta$? Explain.

   (d) Now consider what happens if the wage is fixed at $\bar{w}$ corresponding to the initial z and β. How if at all do your answers to (c) change if the wage is kept fixed at at $\bar{w}$? Explain.

   SOLUTIONS:

   (a) The Bellman equations for a firm are

$$J_t = z - w_t + \beta(\, \delta V_{t+1} + (1-\delta) J_{t+1}\,)$$

   and

$$V_t = -\kappa z + \beta(\, q(\theta_t) J_{t+1} + (1 - q(\theta_t)) V_{t+1}\,)$$

   The value of having a filled job J_t is given by the flow profit $z - w_t$ plus with exogenous probability δ the job is destroyed and the firm switches from J_{t+1} to V_{t+1}. Likewise the value of having a vacancy V_t is given by the cost of keeping a vacancy open $-\kappa z$ plus with endogenous probability $q(\theta_t)$ there is a match and the vacancy is filled so that the firm switches from V_{t+1} to J_{t+1}. The corresponding Bellman equations for a worker are

$$W_t = w_t + \beta(\, \delta U_{t+1} + (1-\delta) W_{t+1}\,)$$

   and

$$U_t = b + \beta(\, f(\theta_t) W_{t+1} + (1 - f(\theta_t)) U_{t+1}\,)$$

The value of having a job W_t is given by the wage w_t plus with exogenous probability δ the job is destroyed and the worker switches from W_{t+1} to U_{t+1}. Likewise the value of being unemployed U_t is given by the unemployment benefit b plus with endogenous probability $f(\theta_t)$ there is a match and the worker switches from U_{t+1} to W_{t+1}. Note that in these Bellman equations, matches (if any) take place at the end of period t and so are operative at the beginning of period $t+1$.

(c) In steady state we have the Bellman equations for the firm

$$J = z - w + \beta(\,\delta V + (1-\delta)J\,), \qquad \text{and} \qquad V = -\kappa z + \beta(\,q(\theta)J + (1-q(\theta))V\,)$$

and for the worker

$$W = w + \beta(\,\delta U + (1-\delta)W\,), \qquad \text{and} \qquad U = b + \beta(\,f(\theta)W + (1-f(\theta))U\,)$$

And from Nash-Bargaining

$$W - U = \frac{\phi}{1-\phi}(J - V)$$

The firm's Bellman equation for a filled job gives

$$J = \frac{1}{1-\beta(1-\delta)}\big(\,z - w + \beta\delta V\,\big)$$

and since $V = 0$ from free-entry, we also have

$$J = \frac{\kappa z}{\beta q(\theta)}$$

Eliminating J between these equations and rearranging gives

$$\boxed{w = z - (1-\beta(1-\delta))\frac{\kappa z}{\beta q(\theta)}} \tag{1}$$

The wage is the marginal productivity of the worker z less the costs of hiring through the frictional labor market. This 'marginal productivity condition' is a downward sloping relationship between θ and w (since $q(\theta)$ is decreasing in θ). With the given matching function $M(u,v) = u^{\alpha}v^{1-\alpha}$ we have $q = M(u,v)/v$ so that $q(\theta) = \theta^{-\alpha}$ and $f = M(u,v)/u$ so that $f(\theta) = \theta^{1-\alpha}$. Given this, we can write the marginal productivity condition as

$$w = z - (1-\beta(1-\delta))\frac{\kappa z}{\beta}\theta^{\alpha}$$

Turning now to the worker side of things, first write

$$W = \frac{1}{1-\beta(1-\delta)}\big(\,w + \beta\delta U\,\big), \qquad J = \frac{1}{1-\beta(1-\delta)}\big(\,z - w\,\big)$$

Then given surplus splitting $W - U = \phi(W - U + J)$ this becomes

$$w - (1-\beta)U = \phi\big(\,w - (1-\beta)U + z - w\,\big)$$

Collecting terms and simplifying

$$w = \phi z + (1-\phi)(1-\beta)U$$

Wage is bargaining-weighted average of productivity z and flow value of unemployment $(1-\beta)U$. Then from the Bellman equation for U we have

$$(1-\beta)U = b + \beta f(\theta)(W - U)$$

And from Nash-Bargaining the worker surplus is proportional to the firm surplus which is pinned down by free entry

$$W - U = \frac{\phi}{1-\phi} J = \frac{\phi}{1-\phi}\left(\frac{\kappa z}{\beta q(\theta)}\right)$$

Hence

$$(1-\beta)U = b + \beta f(\theta)\frac{\phi}{1-\phi}\left(\frac{\kappa z}{\beta q(\theta)}\right) = b + \frac{\phi}{1-\phi}\kappa z\theta$$

Plugging this into our expression for wages and collecting terms

$$\boxed{w = (1-\phi)b + \phi(1+\kappa\theta)z} \tag{2}$$

This is the 'wage curve,' an upward sloping relationship between θ and w. Together, the wage curve and the marginal productivity condition are two equations that we can solve for w, θ in terms of the parameters. Given labor market tightness determined in this way, we can then back out the unemployment rate u from the Beveridge curve

$$u = \frac{\delta}{\delta + f(\theta)} = \frac{\delta}{\delta + \theta^{1-\alpha}}$$

and back out vacancies v from $v = \theta u$.

(d) (Sketch) With higher productivity the wage w rises and labor market tightness θ increases. This is because although the wage curve shifts up and the marginal productivity condition shifts out (with offsetting implications for labor market tightness), you can nonetheless show that the effect on the marginal productivity condition is *larger* so that on net θ rises. Both effects drive wages higher. With θ higher the job finding rate $f(\theta) = \theta^{1-\alpha}$ is also higher and so steady state unemployment $u = \delta/(\delta + f(\theta))$ is lower as we rotate counter-clockwise along the Beveridge curve. Vacancies $v = \theta u$ are also higher and the vacancy filling rate $q(\theta)$ is correspondingly lower.

With a lower discount factor the wage w is lower as is labor market tightness θ. This is because, by making job creation more expensive (in discounted terms), the decrease in β shifts the marginal productivity condition down along an unchanged wage curve thereby reducing both w and θ. With θ lower the job finding rate $f(\theta)$ is also lower and so steady state unemployment $u = \delta/(\delta + f(\theta))$ is higher as we rotate clockwise along the Beveridge curve. Vacancies $v = \theta u$ are also lower and the vacancy filling rate $q(\theta)$ is correspondingly higher.

(e) Suppose we fix the wage at some constant level $\bar{w}$. Then we no longer have Nash-Bargaining and hence no longer have a wage curve. Instead we simply have the marginal productivity condition

$$\bar{w} = z - (1 - \beta(1-\delta))\frac{\kappa z}{\beta q(\theta)}$$

which we can solve for θ. Now if z increases we get a larger increase in labor market tightness and hence unemployment falls by a larger amount. This is because with the wage fixed at $\bar{w}$ there is no upward sloping wage curve to mitigate the effects of higher productivity on labor market tightness (i.e., there is no feedback from z to w to θ). Since firms keep the extra profits from the higher productivity, the overall incentives for job creation are stronger. By contrast, with Nash-Bargaining the wage would rise, so that workers share in some of the gains associated with higher z.

Similarly, with a lower discount factor — we again get bigger movements. Again the intuition is that without the wage curve, there is no change in w to mitigate the effects of the adverse change in job creation conditions. With the rigid wage, the firm bears most of the brunt of the decrease in β (increase in interest rates) but as a result is less inclined to create jobs and so steady state unemployment is higher. By contrast, with Nash-Bargaining the wage would fall so that workers share in some of the losses associated with lower β.