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ECON90003
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Macroeconomics

Tutorial #5: Solutions

1. **Asset pricing with log utility.** Suppose there is a durable asset that pays dividends y_t that follow a Markov chain with transition probabilities $\pi(y' \,|\, y) = \text{Prob}[y_{t+1} = y' \,|\, y_t = y]$. At the beginning of period t the representative consumer's holdings of this asset are k_t and they choose how many units of this asset to hold for the next period k_{t+1}. Their initial endowment of the asset is normalized to $k_0 = 1$. The price of this asset is p_t and is taken as given. The representative consumer seeks to maximize

$$\mathbb{E}\left\{\sum_{t=0}^{\infty} \beta^t \log(c_t)\right\}, \qquad 0 < \beta < 1 \tag{1}$$

subject to the budget constraints

$$c_t + p_t k_{t+1} \leq (p_t + y_t) k_t$$

along with $c_t \geq 0$ and $k_{t+1} \geq 0$.

(a) Let $v(k, y)$ denote the representative consumer's value function. Setup the representative consumer's dynamic programming problem in terms of this value function and define a recursive competitive equilibrium for this economy.

(b) Let $p(y)$ denote the price of the asset when the current dividend state is y. Show that in equilibrium $p(y)$ solves the functional equation

$$p(y) = \beta \sum_{y'} \frac{y}{y'} \,[p(y') + y']\, \pi(y' \,|\, y) \tag{2}$$

(c) Let $f(y) = p(y)/y$ so that $f(y)$ solves the functional equation

$$f(y) = \beta \sum_{y'} (f(y') + 1)\, \pi(y' \,|\, y)$$

Let T denote the operator on the RHS of this equation

$$Tf(y) \equiv \beta \sum_{y'} (f(y') + 1)\, \pi(y' \,|\, y)$$

Show that T satisfies Blackwell's sufficient conditions for a contraction mapping.

(d) Solve the functional equation (2) for $p(y)$.

SOLUTIONS:

(a) Taking $p(y)$ as given, the Bellman equation for the representative consumer can be written

$$v(k,y) = \max_{k'\geq 0} \Big[\log(\,(p(y)+y)k - p(y)k'\,) + \beta \sum_{y'} v(k',y')\pi(y'\,|\,y) \Big]$$

Let $k' = g(k,y)$ denote the optimal policy that achieves the maximum on the RHS of the Bellman equation. A *recursive competitive equilibrium* is three functions, $v(k,y)$, $g(k,y)$ and $p(y)$ such that (i) taking $p(y)$ as given $v(k,y)$ and $g(k,y)$ solve the representative consumer's dynamic programming problem, and (ii) the asset market clears, $g(k,y) = 1$ for all k, y. If the asset market clears, then so does the goods market, so that $c = y$.

(b) The first order condition for k' can be written

$$\frac{1}{c}\,p(y) = \beta \sum_{y'} v_1(k',y')\pi(y'\,|\,y)$$

where it is understood that c is evaluated at the optimum and where $v_1(k,y)$ denotes the derivate of the value function with respect to its first argument. Using the optimal policy $g(k,y)$ we can write

$$v(k,y) = \log(\,(p(y)+y)k - p(y)g(k,y)\,) + \beta \sum_{y'} v(g(k,y),y')\pi(y'\,|\,y)$$

Hence applying the envelope theorem gives

$$v_1(k,y) = \frac{1}{c}\,(p(y)+y)$$

so that

$$v_1(k',y') = \frac{1}{c'}\,(p(y')+y')$$

Using this expression to eliminate $v_1(k',y')$ from the first order condition gives

$$\frac{1}{c}\,p(y) = \beta \sum_{y'} \frac{1}{c'}\,(p(y')+y')\,\pi(y'\,|\,y)$$

Multiplying both sides by c and using $c = y$ then gives the equilibrium asset pricing equation

$$p(y) = \beta \sum_{y'} \frac{y}{y'}\,(p(y')+y')\,\pi(y'\,|\,y)$$

(c) Blackwell's sufficient conditions require (i), that the operator is monotone, in the sense that if $f_1 \geq f_2$ than $Tf_1 \geq Tf_2$, and (ii) that the operator satisfies the discounting property, $T(f+a) \leq Tf + \beta a$ for some $\beta \in (0,1)$. To see (i), suppose that $f_1 \geq f_2$. Then

$$Tf_1(y) = \beta \sum_{y'} (f_1(y')+1)\,\pi(y'\,|\,y)$$

and

$$Tf_2(y) = \beta \sum_{y'} (f_2(y')+1)\,\pi(y'\,|\,y)$$

So that

$$Tf_1(y) - Tf_2(y) = \beta \sum_{y'} (f_1(y') - f_2(y'))\, \pi(y' \,|\, y) \geq 0$$

Hence indeed

$$Tf_1(y) \geq Tf_2(y)$$

To see (ii), observe

$$T(f+a)(y) = \beta \sum_{y'} (f(y') + a + 1)\, \pi(y' \,|\, y)$$

$$= \beta \sum_{y'} (f(y') + 1)\, \pi(y' \,|\, y) + \beta \sum_{y'} a\, \pi(y' \,|\, y)$$

$$= Tf(y) + \beta a \sum_{y'} \pi(y' \,|\, y)$$

$$= Tf(y) + \beta a$$

since for each y the sum of transition probabilities is $\sum_{y'} \pi(y' \,|\, y) = 1$. Hence T also satisfies the discounting property. Since (i) and (ii) are satisfied, T is a contraction mapping.

(d) Guess that the functional equation $f = Tf$ is solved by a constant function $f(y) = a$ for all y. Such a constant satisfies

$$a = \beta \sum_{y'} (a+1)\, \pi(y' \,|\, y) = \beta(a+1)$$

Hence

$$a = \frac{\beta}{1-\beta}$$

Since $f(y) = a$ and $p(y) = f(y)y$, the solution to (2) is

$$p(y) = \frac{\beta}{1-\beta} y$$

2. **Asset pricing with CRRA utility and IID dividends.** Now suppose that the period utility function in (1) is the more general CRRA specification

$$u(c) = \frac{c^{1-\alpha} - 1}{1-\alpha}, \qquad \alpha > 0$$

and that dividends are IID over time so that $\pi(y' \,|\, y) = \pi(y')$ independent of the current y. Show that in equilibrium $p(y)$ now solves the functional equation

$$p(y) = \beta \sum_{y'} \left(\frac{y'}{y}\right)^{-\alpha} (p(y') + y')\, \pi(y') \tag{3}$$

Solve the functional equation (3) for $p(y)$. Other things equal, are asset prices higher or lower when the representative consumer is more risk averse? Give as much intuition as you can.

Solution:

With this more general CRRA specification the representative consumer's first order condition for k' can be written

$$c^{-\alpha}\, p(y) = \beta \sum_{y'} v_1(k', y') \pi(y' \,|\, y)$$

where it is understood that c is evaluated at the optimum. Applying the envelope theorem gives

$$v_1(k, y) = c^{-\alpha}\, (p(y) + y)$$

Using this expression to eliminate $v_1(k', y')$ from the first order condition gives

$$c^{-\alpha}\, p(y) = \beta \sum_{y'} c'^{-\alpha}\, (p(y') + y')\, \pi(y' \,|\, y)$$

Multiplying both sides by c^{α} and using $c = y$ then gives the equilibrium asset pricing equation

$$p(y) = \beta \sum_{y'} \left(\frac{y'}{y}\right)^{-\alpha} (p(y') + y')\, \pi(y' \,|\, y)$$

In the further special case that y' is IID so that $\pi(y' \,|\, y) = \pi(y')$ for all y we have

$$p(y) = \beta \sum_{y'} \left(\frac{y'}{y}\right)^{-\alpha} (p(y') + y')\, \pi(y')$$

To solve this functional equation, let $f(y) = p(y)/y^{\alpha}$ so that

$$f(y) = \beta \sum_{y'} (f(y') + {y'}^{1-\alpha})\, \pi(y')$$

Now guess that this functional equation is solved by a constant function $f(y) = a$ for all y. Such a constant satisfies

$$a = \beta \sum_{y'} (a + {y'}^{1-\alpha})\, \pi(y') = \beta a + \beta \delta$$

where

$$\delta \equiv \sum_{y'} {y'}^{1-\alpha}\, \pi(y') = \mathbb{E}\{y^{1-\alpha}\}$$

Hence

$$a = \frac{\beta \delta}{1-\beta}$$

Since $f(y) = a$ and $p(y) = f(y) y^{\alpha}$, the solution is

$$p(y) = \frac{\beta \delta}{1-\beta} y^{\alpha}, \qquad \delta = \mathbb{E}\{y^{1-\alpha}\}$$

Notice that in the log utility case, $\alpha = 1$, we have $\delta = 1$ so this agrees with our calculation in question 1 above (which, however, did not impose that dividends are IID).