Faculty of Business and Economics
Department of Economics
THE UNIVERSITY OF MELBOURNE

ECON90003
Chris Edmond

Macroeconomics
Tutorial #4: Solutions

1. **An exactly solved Bellman equation, revisited.** Consider the problem of maximizing

$$\mathbb{E}\left\{\sum_{t=0}^{\infty}\beta^t\log(c_t)\right\}, \qquad 0<\beta<1 \tag{1}$$

subject to the constraints

$$c_t\geq 0, \qquad k_{t+1}\geq 0, \qquad c_t+k_{t+1}\leq z_t k_t^{\alpha}, \qquad 0<\alpha<1, \qquad t=0,1,\ldots$$

for given initial condition $k_0>0$ and some given stochastic process $\{z_t\}$.

Let $v(k,z)$ denote the value function for this problem. The value function $v(k,z)$ solves the Bellman equation

$$v(k,z)=\max_x\Big\{\log(zk^{\alpha}-x)+\beta\mathbb{E}[v(x,z')\,|\,z]\Big\}$$

Verify that the solution for $v(k,z)$ is of the form

$$v(k,z)=A+B\log k+C\log z$$

where the coefficients are given by

$$A\equiv\frac{1}{1-\beta}\left(\log(1-\alpha\beta)+\frac{\alpha\beta}{1-\alpha\beta}\log(\alpha\beta)+\frac{\beta}{1-\alpha\beta}\mathbb{E}\{\log z'|z\}\right)$$

and

$$B\equiv\frac{\alpha}{1-\alpha\beta}>\alpha>0$$

and

$$C\equiv\frac{1}{1-\alpha\beta}>1$$

Solution:

The Bellman equation for this problem is

$$v(k,z)=\max_x\Big\{\log(zk^{\alpha}-x)+\beta\mathbb{E}[v(x,z')\,|\,z]\Big\}$$

Guess that $v(k,z)=A(z)+B\log k+C\log z$ solves the Bellman equation. Then the RHS of the Bellman equation is

$$\max_x\Big\{\log(zk^{\alpha}-x)+\beta\big(A(z)+B\log x+C\mathbb{E}[\log z'\,|\,z]\big)\Big\}$$

Notice that the expectation over the productivity shock next period z' enters in a completely separable way and so does not affect the optimal choice of x. The first order condition for this problem can be written

$$\frac{1}{zk^\alpha - x} = \beta\frac{B}{x}$$

which implies the optimal policy

$$x = \frac{\beta B}{1+\beta B} zk^\alpha \equiv g(k,z)$$

Plugging this back into the RHS of the Bellman equation gives

$$v(k,z) = \log\left(zk^\alpha - \frac{\beta B}{1+\beta B}zk^\alpha\right) + \beta\left(A + B\log\left(\frac{\beta B}{1+\beta B}zk^\alpha\right) + C\mathbb{E}[\log z' \,|\, z]\right)$$

$$= \log\left(\frac{1}{1+\beta B}zk^\alpha\right) + \beta A + \beta B\log\left(\frac{\beta B}{1+\beta B}zk^\alpha\right) + \beta C\mathbb{E}[\log z' \,|\, z]$$

which is indeed of the form

$$v(k,z) = A + B\log k + C\log z$$

Matching the slope coefficients we get

$$B = \alpha + \alpha\beta B$$

and

$$C = 1 + \beta B$$

which gives

$$B = \frac{\alpha}{1-\alpha\beta} > \alpha > 0$$

and

$$C = 1 + \beta\frac{\alpha}{1-\alpha\beta} = \frac{1}{1-\alpha\beta} > 1$$

And finally matching the intercept we get

$$A = \log\left(\frac{1}{1+\beta B}\right) + \beta A + \beta B\log\left(\frac{\beta B}{1+\beta B}\right) + \beta C\mathbb{E}[\log z' \,|\, z]$$

Plugging in our solutions for B and C and solving for A then gives

$$A = \frac{1}{1-\beta}\left(\log(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta}\log(\alpha\beta) + \frac{\beta}{1-\alpha\beta}\mathbb{E}\{\log z'|z\}\right)$$

2. **Stationary distribution with lognormal shocks.** Suppose that the shocks z_t are IID lognormal

$$\log(z_t) \sim N(\mu, \sigma^2)$$

Let $\mu_{k,t}$ and $\sigma^2_{k,t}$ denote the mean and variance of the time-t distribution of the log capital stock $\log(k_t)$. Using (2), solve for the sequence of means $\mu_{k,t}$ and variances $\sigma^2_{k,t}$ and calculate their limiting values as $t \to \infty$. What is the stationary distribution of the log capital stock?

Solution:

Under the policy in (2) we have

$$\log k_{t+1} = \log(\alpha\beta) + \alpha \log k_t + \log z_t$$

So that the log capital stock is an AR(1) process with IID normal innovations.

Means. For $t = 1, 2, \ldots$ we have

$$\mu_{k,t} \equiv \mathbb{E}_0[\log k_t] = \log(\alpha\beta) + \alpha\mathbb{E}_0[\log k_{t-1}] + \mathbb{E}_0[\log z_{t-1}] \tag{3}$$

Hence for $t = 1$

$$\mu_{k,1} = \log(\alpha\beta) + \alpha \log k_0 + \log z_0$$

Now define the initial condition

$$\mu_{k,0} \equiv \log k_0 + \frac{1}{\alpha}(\log z_0 - \mu)$$

So that we can equivalently write for $t = 1$

$$\mu_{k,1} = \log(\alpha\beta) + \mu + \alpha\mu_{k,0} \tag{3}$$

thereby revealing that the means follow a deterministic difference equation. Iterating forward

$$\begin{aligned}\mu_{k,2} &= \log(\alpha\beta) + \mu + \alpha\mu_{k,1} \\ &= \log(\alpha\beta) + \mu + \alpha(\log(\alpha\beta) + \mu + \alpha\mu_{k,0}) \\ &= (\log(\alpha\beta) + \mu)(1 + \alpha) + \alpha^2\mu_{k,0}\end{aligned}$$

And doing this to period t gives

$$\begin{aligned}\mu_{k,t} &= (\log(\alpha\beta) + \mu)\sum_{j=0}^{t-1}\alpha^j + \alpha^t\mu_{k,0} \\ &= (\log(\alpha\beta) + \mu)\frac{1-\alpha^t}{1-\alpha} + \alpha^t\mu_{k,0}\end{aligned}$$

Which is of the form

$$\mu_{k,t} = \mu_k^* + \alpha^t\big(\mu_{k,0} - \mu_k^*\big)$$

where

$$\mu_k^* \equiv \frac{\log(\alpha\beta) + \mu}{1-\alpha}$$

So in the limit

$$\lim_{t\to+\infty} \mu_{k,t} = \mu_k^*$$

In the long-run, the sequence of means converges to μ_k^*. Notice that this is the unique fixed point of the difference equation (3) above.

Variances. Similarly, for $t = 1, 2, \ldots$ we have

$$\begin{aligned} \sigma_{k,t}^2 \equiv \text{Var}_0[\log k_t] &= 0 + \alpha^2 \text{Var}_0[\log k_{t-1}] + \text{Var}_0[\log z_{t-1}] \\ &= \alpha^2 \sigma_{k,t-1}^2 + \sigma^2 \end{aligned} \tag{4}$$

Thus the variances also follow a deterministic difference equation. Following the usual steps we get the solution

$$\sigma_{k,t}^2 = \sigma_k^{2\,*} + \alpha^{2(t-1)} \big(\sigma_{k,1}^2 - \sigma_k^{2\,*} \big), \qquad t = 2, 3, \ldots$$

where

$$\sigma_{k,1}^2 = \text{Var}_0[\log k_1] = \text{Var}_0[\log(\alpha\beta) + \alpha \log k_0 + \log z_0] = 0$$

and where

$$\sigma_k^{2\,*} \equiv \frac{\sigma^2}{1 - \alpha^2}$$

is the unique fixed point of (4) and in the limit

$$\lim_{t \to +\infty} \sigma_{k,t}^2 = \sigma_k^{2\,*}$$

So that in the long-run the sequence of variances converges to $\sigma_k^{2\,*}$. To summarize, for each t the distribution of capital is lognormal with

$$\log k_t \sim N\big(\mu_{k,t} \, , \, \sigma_{k,t}^2 \, \big)$$

And this sequence of distributions converges over time to a limiting lognormal distribution

$$\log k \sim N\big(\mu_k^* \, , \, \sigma_k^{2\,*} \, \big)$$

with moments

$$\mu_k^* = \frac{\log(\alpha\beta) + \mu}{1 - \alpha}, \qquad \text{and} \qquad \sigma_k^{2\,*} = \frac{\sigma^2}{1 - \alpha^2}$$