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ECON90003
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Macroeconomics
Tutorial #3: Solutions

1. **Cake-eating.** Consider the problem of choosing consumption c_t for $t = 0, 1, \ldots$ to maximize

$$\sum_{t=0}^{\infty} \beta^t u(c_t), \qquad 0 < \beta < 1 \tag{1}$$

subject to the constraints

$$c_t \geq 0, \qquad k_{t+1} \geq 0, \qquad c_t = k_t - k_{t+1}, \qquad t = 0, 1, \ldots$$

with given initial condition

$$k_0 > 0$$

Suppose that period utility has the isoelastic form

$$u(c) = \frac{c^{1-\sigma} - 1}{1 - \sigma}, \qquad \sigma > 0$$

(a) Consider a recursive formulation of this problem and let $v(k)$ denote the value function and let $c(k)$ denote the consumption policy function. Setup and explain the Bellman equation that determines these functions.

(b) Solve for $c(k)$. Using this solution, explain the time paths of c_t and k_t starting from the given initial condition k_0. Explain how your answers depend on the parameters β and σ. Give intuition for your results.

Solutions:

(a) The Bellman equation for this problem can be written

$$v(k) = \max_x \left[\frac{(k - x)^{1-\sigma} - 1}{1 - \sigma} + \beta v(x) \right]$$

Supposing we have found a $v(k)$ that solves the Bellman equation, the consumption policy function can be recovered as

$$c(k) = \operatorname*{argmax}_c \left[\frac{c^{1-\sigma} - 1}{1 - \sigma} + \beta v(k - c) \right]$$

(b) The first order condition associated with the RHS of the Bellman equation is

$$(k - x)^{-\sigma} = \beta v'(x)$$

Let $x = g(k)$ solve this first order condition

$$(k - g(k))^{-\sigma} = \beta v'(g(k))$$

The envelope condition gives

$$v'(k) = (k - g(k))^{-\sigma}$$

Evaluating this at $g(k)$ gives

$$v'(g(k)) = (g(k) - g(g(k)))^{-\sigma}$$

Hence the Euler equation for this problem is

$$(k - g(k))^{-\sigma} = \beta(g(k) - g(g(k)))^{-\sigma}, \qquad \text{for all } k$$

This is a functional equation to be solved for $g(k)$. Let's guess that $g(k) = \theta k$ solves this problem for some $\theta \in (0, 1)$. For this guess to be valid, we need

$$(1 - \theta)^{-\sigma} k^{-\sigma} = \beta(\theta - \theta^2)^{-\sigma} k^{-\sigma}, \qquad \text{for all } k$$

which simplifies to

$$1 = \beta\theta^{-\sigma}$$

or

$$\theta = \beta^{1/\sigma}$$

Then since $c(k) = k - g(k)$ we have the consumption policy function

$$c(k) = (1 - \beta^{1/\sigma})\, k$$

Notice that $\theta = \beta^{1/\sigma}$ is indeed in $(0, 1)$ since $\beta \in (0, 1)$ and $\sigma > 0$. In short each period a fraction $1 - \theta$ of the state k is consumed with the remaining fraction θ left unconsumed and available for consumption next period.

Going back to the sequence notation we then have $k_{t+1} = \theta k_t$ so that

$$k_t = \theta^t\, k_0 = \beta^{t/\sigma}\, k_0$$

and hence

$$c_t = (1 - \theta)k_t = (1 - \beta^{1/\sigma})\beta^{t/\sigma}\, k_0$$

Think of the state k as an infinitely durable cake with initial size $k_0 > 0$. Each period the consumer eats a fraction $1 - \theta$ of the cake and leaves θ of the cake for the next period. Over time the cake is shrinking geometrically as $\beta^{t/\sigma} \to 0$. The rate at which the cake is eaten is high if β is close to zero (the consumer is impatient) or if σ is low (the consumer is very willing to substitute consumption over time). The rate at which the cake is eaten is low if β is close to one (the consumer is patient) or if σ is high (the consumer is very unwilling to substitute consumption over time).

2. **Borrowing and lending.** Consider a risk neutral consumer choosing consumption c_t for $t = 0, 1, \ldots$ to maximize

$$\sum_{t=0}^{\infty} \beta^t c_t, \qquad 0 < \beta < 1 \tag{2}$$

The consumer can borrow or lend freely at gross interest rate $R = 1/\beta$ and so faces the sequence of constraints

$$0 \le c_t \le a_t - \beta a_{t+1}, \qquad t = 0, 1, \ldots$$

with given initial wealth a_0. The interpretation $a_t > 0$ here means the consumer is a net lender but $a_t < 0$ means the consumer is a net borrower. To begin with, suppose that there are no constraints on borrowing so that $a_t < 0$ is permitted.

(a) Let $v^*(a_0)$ denote the solution of the sequence problem (maximizing (2) subject to the sequence of constraints $0 \le c_t \le a_t - \beta a_{t+1}$) and let $v(a)$ denote the value function that solves the associated Bellman equation. Do the solutions coincide, i.e., does $v(a_0) = v^*(a_0)$? Why or why not?

(b) Now suppose that we prohibit borrowing, that is we add the constraint $a_{t+1} \ge 0$. How if at all does this change your answer to (a)?

Solutions:

(a) Since the consumer can borrow without limit, consumption is unbounded and the sequence approach gives $v^*(a_0) = +\infty$ for all a_0. For the recursive approach, the Bellman equation is

$$v(a) = \max_{\beta a' \le a} \Big[a - \beta a' + \beta v(a') \Big]$$

Notice that $v^*(a) = +\infty$ solves the Bellman equation

$$+\infty = v^*(a) = \max_{\beta a' \le a} \Big[a - \beta a' + \beta v^*(a') \Big] = \max_{\beta a' \le a} \Big[a - \beta a' + \infty \Big] = +\infty$$

But $v(a) = a$ also solves the Bellman equation

$$a = v(a) = \max_{\beta a' \le a} \Big[a - \beta a' + \beta v(a) \Big] = \max_{\beta a' \le a} \Big[a - \beta a' + \beta a' \Big] = a$$

Now recall the boundedness condition that we need to be sure that solutions to the recursive problem are also solutions to the original sequence problem

$$\lim_{t \to +\infty} \beta^t v(a_t) = 0$$

The value function $v(a) = a$ implies the policy $a' = g(a) = a/\beta$ hence $v(a_t) = a_t = \beta^{-t} a_0$ and hence

$$\lim_{t \to +\infty} \beta^t v(a_t) = \lim_{t \to +\infty} \beta^t \beta^{-t} a_0 = a_0 > 0$$

Thus this solution violates the boundedness condition and hence does not solve the sequence problem. The correct solution to the sequence problem involves borrowing without limit.

(b) If we also had the borrowing constraint $a_{t+1} \geq 0$ then $c_t = a_t - \beta a_{t+1}$ with $c_t \in [0, a_t]$ (bounded) and by direct calculation

$$v^*(a_0) = \sum_{t=0}^{\infty} \beta^t (a_t - \beta a_{t+1})$$

$$= (a_0 - \beta a_1) + \beta(a_1 - \beta a_2) + \beta^2(a_2 - \beta a_3) + \dots$$

$$= a_0$$

Hence we now have $v^*(a) = a$. Moreover for the recursive approach, the Bellman equation is

$$v(a) = \max_{0 \leq \beta a' \leq a} \Big[a - \beta a' + \beta v(a') \Big]$$

Which is solved by $v(a) = a$

$$a = v(a) = \max_{0 \leq \beta a' \leq a} \Big[a - \beta a' + \beta v(a') \Big] = \max_{0 \leq \beta a' \leq a} \Big[a - \beta a' + \beta a' \Big] = a$$

Hence now the recursive approach and the sequential approach yield the same value function. Notice that for any fixed period t the consumer is indifferent between any $a_{t+1} \in [0, a_t/\beta]$. But also notice that setting $a_{t+1} = a_t/\beta$ so that $a_t = \beta^{-t} a_0$ does not solve the consumer's problem because it implies $c_t = 0$ for all t. While the consumer is indifferent amongst $a_{t+1} \in [0, a_t/\beta]$ for any fixed t, they must at some point choose to consume something. Put differently, the consumer views consumption on different dates as perfect substitutes, so, as long as the consumer eventually consumes something (in finite time) they will get the payoff $v(a_0)$ depending on their initial a_0.