ECON90003
Chris Edmond

Macroeconomics
Tutorial #2

1. **An exactly-solved Bellman equation.** Consider a discrete-time infinite-horizon optimal growth model where the planner chooses capital stocks k_{t+1} for $t = 0, 1, \ldots, T$ to maximize

$$\sum_{t=0}^{\infty} \beta^t \log c_t, \qquad 0 < \beta < 1 \tag{1}$$

subject to the sequence of resource constraints

$$c_t + k_{t+1} \leq k_t^{\alpha}, \qquad t = 0, 1, \ldots$$

with given initial condition

$$k_0 > 0$$

Let $v(k)$ denote the value function for this problem. The value function $v(k)$ solves the Bellman equation

$$v(k) = \max_x \Big[\log(k^{\alpha} - x) + \beta v(x) \Big]$$

Verify that the solution for $v(k)$ is

$$v(k) = A + B \log k$$

where

$$A = \frac{1}{1-\beta}\left(\log(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta}\log(\alpha\beta)\right)$$

and

$$B = \frac{\alpha}{1-\alpha\beta} > 0$$

Solutions:

Guess that $v(k) = A + B\log k$ solves the Bellman equation. Then the RHS of the Bellman equation is

$$\max_{0 \leq x \leq k^{\alpha}} \Big[\log(k^{\alpha} - x) + \beta(A + B\log x) \Big]$$

This is a standard concave optimization problem and its solution is completely characterized by the first order condition with respect to the choice variable x

$$\frac{1}{k^{\alpha} - x}(-1) + \beta\frac{B}{x} = 0$$

which solves the optimal policy

$$x = \frac{\beta B}{1 + \beta B} k^{\alpha} \equiv g(k)$$

Plugging this back into the RHS of the Bellman equation gives

$$v(k) = \log\left(k^\alpha - \frac{\beta B}{1+\beta B}k^\alpha\right) + \beta\left(A + B\log\left(\frac{\beta B}{1+\beta B}k^\alpha\right)\right)$$

$$= \log\left(\frac{1}{1+\beta B}k^\alpha\right) + \beta A + \beta B\log\left(\frac{\beta B}{1+\beta B}k^\alpha\right)$$

$$= \beta A + \beta B\log(\beta B) - (1+\beta B)\log(1+\beta B) + (1+\beta B)\alpha\log k$$

Now observe that matching coefficients we have

$$(1+\beta B)\alpha = B$$

and

$$\beta A + \beta B\log(\beta B) - (1+\beta B)\log(1+\beta B) = A$$

We can then solve the first of these for B to get,

$$B = \frac{\alpha}{1-\alpha\beta} > 0$$

Then using this to solve for A gives

$$A = \frac{1}{1-\beta}\left(\beta B\log(\beta B) - (1+\beta B)\log(1+\beta B)\right)$$

$$= \frac{1}{1-\beta}\left(\log(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta}\log(\alpha\beta)\right)$$

as required.

2. **Properties of the policy function.** Consider a strictly increasing strictly concave production function $f(k)$ that satisfies $f(0) = 0$ and $f'(0) = +\infty$ and $f'(\infty) < 1$. Suppose the policy function $k_{t+1} = g(k_t)$ has the form $g(k) = sf(k)$ for some $s \in (0,1)$.

   (a) Show that there is exactly one steady state $k^* > 0$.

   (b) Show that for any $k_0 > 0$ the sequence $\{k_{t+1}\}_{t=0}^{\infty}$ induced by $g(k)$ converges monotonically to k^*.

   SOLUTIONS:

   (a) Since $g(k) = sf(k)$ with $s \in (0,1)$ we have $g(0) = s0 = 0$ and $g'(k) = sf'(k) > 0$ and $g''(k) = sf''(k) < 0$ with $g'(0) = sf'(0) = +\infty$ and $g'(\infty) = sf'(\infty) < 1$ since $s < 1$ and $f'(\infty) < 1$. In short, $g(k)$ inherits the strictly increasing strictly concave shape of $f(k)$. A steady state $k^* > 0$ solves

   $$k^* = g(k^*)$$

Let's write this as

$$h(k^*) = 0, \qquad \text{where} \qquad h(k) \equiv \frac{g(k)}{k} - 1 = s\frac{f(k)}{k} - 1$$

with

$$h'(k) = s\left(\frac{f'(k)}{k} - \frac{f(k)}{k^2}\right) \leq 0$$

Note: to see why $h'(k) \leq 0$, note that a concave function f is bounded above by its first-order approximation so that for any x, y we have

$$f(y) \leq f(x) + f'(x)(y - x)$$

(the first order approximation is a *global over-estimator* of f; this is the key to using local properties of the function to deduce global properties like its maximum). Now take f to be our production function and set $y = 0$ and $x = k$ to write

$$f(0) \leq f(k) - f'(k)k$$

and since $f(0) = 0$ this implies $f'(k)k \leq f(k)$ and since $k > 0$ this implies $f'(k)/k \leq f(k)/k^2$ hence $h'(k) \leq 0$ (actually strictly so, since f is strictly concave).
Moreover note that by l'Hôpital's rule we have

$$h(0) = s\lim_{k\to 0}\frac{f(k)}{k} - 1 = s\frac{f'(0)}{1} - 1 = +\infty$$

and

$$h(\infty) = s\lim_{k\to\infty}\frac{f(k)}{k} - 1 = s\frac{f'(\infty)}{1} - 1 < 0$$

(since $sf'(\infty) < f'(\infty) < 1$). Hence the function $h(k)$ is continuous and monotonically decreasing on $[0, \infty)$ and moreover there exists constants $\underline{k} < \overline{k}$ such that $h(k) > 0$ for all $k < \underline{k}$ and $h(k) < 0$ for all $k > \overline{k}$. Then by the intermediate value theorem there is a unique $k^* \in [\underline{k}, \overline{k}]$ such that $h(k^*) = 0$ and hence for this k^* we have $g(k^*) = sf(k^*) = k^*$ so that k^* is the unique steady state.

(b) Notice that

$$\begin{aligned} k_{t+1} > k_t \quad &\Leftrightarrow \quad g(k_t) > k_t \\ &\Leftrightarrow \quad sf(k_t) > k_t \\ &\Leftrightarrow \quad h(k_t) > 0 \\ &\Leftrightarrow \quad k_t < k^* \end{aligned}$$

so that the sequence $\{k_{t+1}\}_{t=0}^{\infty}$ induced by $g(k)$ is strictly monotonically increasing if it starts from some $k_0 < k^*$. Since this sequence is strictly monotonically increasing and bounded above by k^* the sequence converges to k^* (from below) as $t \to \infty$. Likewise the sequence $\{k_{t+1}\}_{t=0}^{\infty}$ induced by $g(k)$ is strictly monotonically decreasing if it starts from some $k_0 > k^*$. And since this sequence is strictly monotonically decreasing and bounded below by k^* the sequence converges to k^* (from above) as $t \to \infty$. Hence in either case the convergence is monotone. Of course if $k_0 = k^*$ then we are immediately at steady state.