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ECON90003
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Macroeconomics
Tutorial #1: Solutions

1. To be done in class.

2. **Finite-horizon optimal growth model.** Consider a discrete-time finite-horizon optimal growth model where the planner chooses capital stocks k_{t+1} for $t = 0, 1, \ldots, T$ to maximize

$$\sum_{t=0}^{T} \beta^t u(c_t), \qquad 0 < \beta < 1$$

subject to the sequence of resource constraints

$$c_t + k_{t+1} \leq f(k_t), \qquad t = 0, 1, \ldots, T$$

with given initial condition

$$k_0 > 0$$

To begin with, assume the utility and production functions satisfy $u'(c) > 0$, $u''(c) < 0$ and $f'(k) > 0$, $f''(k) < 0$ with $u'(0) = f'(0) = +\infty$ and $f(0) = 0$.

(a) Show that the solution to this problem is characterized by the sequence of conditions

$$u'(f(k_t) - k_{t+1}) = \beta\, u'(f(k_{t+1}) - k_{t+2})\, f'(k_{t+1}), \qquad t = 0, 1, \ldots, T-1 \qquad (*)$$

along with

$$k_{T+1} = 0$$

Explain how these conditions pin down the sequence of capital stocks that solve the planning problem given the initial $k_0 > 0$.

Now suppose in particular that $u(c) = \log c$ and $f(k) = k^\alpha$ for $0 < \alpha < 1$.

(b) Show that the sequence of capital stocks

$$k_{t+1} = \alpha\beta\, \frac{1 - (\alpha\beta)^{T-t}}{1 - (\alpha\beta)^{T-t+1}}\, k_t^\alpha, \qquad t = 0, 1, \ldots, T$$

satisfies the optimality conditions in part (a) above.

(c) Consider the limit $T \to \infty$. Show that in this limit we have

$$k_{t+1} = \alpha\beta\, k_t^\alpha, \qquad c_t = (1 - \alpha\beta)\, k_t^\alpha$$

for given $k_0 > 0$. Interpret these formulas in terms of the usual phase diagram for the discrete-time infinite-horizon optimal growth model.

Solutions:

(a) The Lagrangian for this problem can be written

$$\mathcal{L} = \sum_{t=0}^{T} \beta^t u(c_t) + \sum_{t=0}^{T} \lambda_t [f(k_t) - c_t - k_{t+1}]$$

Since $u(c)$ and $f(k)$ are strictly concave, this is a well-behaved finite dimensional optimization problem and its solution is completely characterized by the Kuhn-Tucker conditions. The key first order conditions for this problem are, for consumption,

$$c_t : \qquad \beta^t u'(c_t) - \lambda_t \leq 0, \qquad t = 0, 1, \dots, T$$

(with strict equality whenever $c_t > 0$), and for capital,

$$k_{t+1} : \qquad -\lambda_t + \lambda_{t+1} f'(k_{t+1}) \leq 0, \qquad t = 0, 1, \dots, T$$

(with strict equality whenever $k_{t+1} > 0$). Likewise for the multipliers

$$\lambda_t : \qquad f(k_t) - c_t - k_{t+1} \geq 0, \qquad t = 0, 1, \dots, T$$

Since $u'(c) > 0$ for all c the resource constraint will always bind so that $c_t = f(k_t) - k_{t+1}$ for all $t = 0, 1, \dots, T$ (all resources are used) and the associated multipliers λ_t are strictly positive. Likewise, since $u'(0) = +\infty$, the first order condition for consumption will always hold with equality

$$\beta^t u'(c_t) = \lambda_t, \qquad t = 0, 1, \dots, T$$

Since $f'(0) = +\infty$, the first order condition for capital will hold with equality whenever λ_{t+1} is positive. We have just seen that λ_t is positive for all $t = 0, 1, ..., T$. But what about λ_{T+1}? Well there is no constraint at all at date $T + 1$ so implicitly $\lambda_{T+1} = 0$. Then since $u'(c) > 0$ the planner choose for all resources at date T to be consumed, $c_T = f(k_T)$ so that $k_{T+1} = 0$. In short we have

$$\lambda_t = \lambda_{t+1} f'(k_{t+1}), \qquad t = 0, 1, \dots, T - 1$$

and

$$k_{T+1} = 0$$

Then since $\lambda_t = \beta^t u'(c_t)$ and $c_t = f(k_t) - k_{t+1}$ for all $t = 0, 1, \dots, T$ we can write

$$u'(f(k_t) - k_{t+1}) = \beta u'(f(k_{t+1}) - k_{t+2}) f'(k_{t+1}), \qquad t = 0, 1, \dots, T - 1$$

which is the Euler equation (∗). This implicitly describes a nonlinear second order difference equation in k_t. In other words, there is family of solutions and to pin down the specific sequence k_t that solves the optimization problem we need two boundary conditions. These are the given initial condition $k_0 > 0$ (which is a parameter of the problem, not a choice variable) and the terminal condition $k_{T+1} = 0$ (which is a necessary condition for an optimum). These two boundary conditions then allow us to pick out the specific solution from the general family of solutions implied by the Euler equation.

(b) With $u(c) = \log c$ and $f(k) = k^\alpha$ the Euler equation $(*)$ becomes

$$\frac{1}{k_t^\alpha - k_{t+1}} = \beta \, \frac{1}{k_{t+1}^\alpha - k_{t+2}} \, \alpha k_{t+1}^{\alpha-1}, \qquad t = 0, 1, \ldots, T-1$$

which can be written as simply

$$\frac{c_{t+1}}{c_t} = \alpha\beta k_{t+1}^{\alpha-1}$$

Using the conjectured law of motion for k_{t+1} we have that

$$c_t = k_t^\alpha - k_{t+1} = \left(1 - \alpha\beta \frac{1-(\alpha\beta)^{T-t}}{1-(\alpha\beta)^{T-t+1}}\right) k_t^\alpha$$

$$= \left(\frac{1-(\alpha\beta)^{T-t+1} - \alpha\beta + \alpha\beta(\alpha\beta)^{T-t}}{1-(\alpha\beta)^{T-t+1}}\right) k_t^\alpha$$

$$= \frac{1-\alpha\beta}{1-(\alpha\beta)^{T-t+1}} \, k_t^\alpha$$

Likewise

$$c_{t+1} = \frac{1-\alpha\beta}{1-(\alpha\beta)^{T-t}} \, k_{t+1}^\alpha$$

Hence

$$\frac{c_{t+1}}{c_t} = \left(\frac{1-(\alpha\beta)^{T-t+1}}{1-(\alpha\beta)^{T-t}}\right) \left(\frac{k_{t+1}}{k_t}\right)^\alpha$$

Hence our law of motion for the capital stock solves the Euler equation when

$$\left(\frac{1-(\alpha\beta)^{T-t+1}}{1-(\alpha\beta)^{T-t}}\right) \left(\frac{k_{t+1}}{k_t}\right)^\alpha = \alpha\beta k_{t+1}^{\alpha-1}$$

Cancelling common terms and rearranging, this requires

$$k_{t+1} = \alpha\beta \, \frac{1-(\alpha\beta)^{T-t}}{1-(\alpha\beta)^{T-t+1}} \, k_t^\alpha$$

Hence the conjectured law of motion for the capital stock indeed solves the Euler equation. Finally, notice that for $t = T$ we have

$$k_{T+1} = \alpha\beta \, \frac{1-(\alpha\beta)^0}{1-(\alpha\beta)^1} \, k_T^\alpha = 0$$

Hence this conjectured law of motion also respects the terminal condition. In short, this law of motion solves the planner's optimization problem.

(c) Since $(\alpha\beta) \in (0,1)$, in the limit as $T \to \infty$, terms like $(\alpha\beta)^{T-t} \to 0$ (for each t) in the law of motion for the capital stock. In particular, we get

$$k_{t+1} = \alpha\beta \lim_{T\to\infty} \left\{\frac{1-(\alpha\beta)^{T-t}}{1-(\alpha\beta)^{T-t+1}}\right\} k_t^\alpha = \alpha\beta \left\{\frac{1-0}{1-0}\right\} k_t^\alpha = \alpha\beta \, k_t^\alpha$$

Likewise for consumption

$$c_t = \lim_{T\to\infty} \left\{ \frac{1-\alpha\beta}{1-(\alpha\beta)^{T-t+1}} \right\} k_t^\alpha = \left\{ \frac{1-\alpha\beta}{1-0} \right\} k_t^\alpha = (1-\alpha\beta)\, k_t^\alpha \equiv c(k_t)$$

In terms of our usual phase diagram, this function $c(k_t)$ is the stable arm of the saddle path and $c_0 = c(k_0)$ is the initial jump that consumption makes to put the economy on a trajectory that converges to steady state. Notice that this system is exactly log-linear

$$\log k_{t+1} = \log(\alpha\beta) + \alpha \log k_t$$

So the coefficient $\alpha \in (0, 1)$ here corresponds to the stable eigenvalue.