Faculty of Business and Economics
Department of Economics

THE UNIVERSITY OF MELBOURNE

ECON90003
Chris Edmond

Macroeconomics
Tutorial #10: Solutions

1. **Static Hopenhayn.** Firms discount flow profits according to a constant discount factor $0 < \beta < 1$. There is an unlimited number of potential entrants. On paying an entry cost $k_e > 0$ an entrant receives a once-and-for-all productivity draw $z \sim g(z)$ and then makes a once-and-for-all decision to operate or exit. On paying a fixed operating cost $k > 0$, a firm that hires n workers can produce output $y = zn^\alpha$ for $0 < \alpha < 1$. Let w denote the wage and p the price of of their output. Let $w = 1$ be the numeraire.

   (a) Let $n(z;p)$, $y(z;p)$ and $\pi(z;p)$ denote the optimal employment policy, output, and profits of a firm with productivity z when the price is p. Solve for these functions. Let $z^*(p)$ denote the lowest level of productivity such that a firm does not exit. Solve for $z^*(p)$. How does z^* depend on p? Explain.

   (b) Let $v(z;p)$ denote the value function of a firm. Solve for $v(z;p)$.

   (c) Use the free-entry condition and the cutoff productivity condition to derive the comparative statics of z^* and p^* with respect to k, k_e and α. Give intuition for your results.

   Now suppose that productivity is drawn from the *Pareto distribution* with density

$$g(z) = \xi z^{-\xi-1}, \qquad z \geq 1, \qquad \xi > 1$$

   (d) Solve explicitly for z^* and p^*. How do z^* and p^* depend on the shape parameter ξ? What is the productivity distribution of actively producing firms? Explain.

   SOLUTIONS:

   (a) The profit maximization problem for a firm of type z facing competitive price p is

$$\pi(z;p) \equiv \max_{n \geq 0} \left[pzn^\alpha - n - k\right]$$

   (note that $w = 1$ is the numeraire). The first order condition for this problem is

$$\alpha pzn^{\alpha-1} = 1$$

   which equates the value of the marginal product of labor to its factor cost. This solves for the optimal employment policy

$$n(z;p) = (\alpha pz)^{\frac{1}{1-\alpha}}$$

which is increasing and convex in z. Plugging this back into the profit function

$$\pi(z;p) = pzn(z;p)^\alpha - n(z;p) - k$$

$$= pz\,(\alpha pz)^{\frac{\alpha}{1-\alpha}} - (\alpha pz)^{\frac{1}{1-\alpha}} - k$$

$$= (1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}\,(pz)^{\frac{1}{1-\alpha}} - k$$

which is likewise increasing and convex in z. The associated level of output is

$$y(z;p) = zn(z;p)^\alpha = z\,(\alpha pz)^{\frac{\alpha}{1-\alpha}} = (\alpha p)^{\frac{\alpha}{1-\alpha}}\; z^{\frac{1}{1-\alpha}}$$

The cutoff productivity $z^*(p)$ is determined by

$$\pi(z^*;p) = 0$$

or equivalently

$$(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}\,(pz^*)^{\frac{1}{1-\alpha}} - k = 0$$

This solves for

$$z^*(p) = \left(\frac{k^{1-\alpha}}{\alpha^\alpha(1-\alpha)^{1-\alpha}}\right)\frac{1}{p}$$

In short, the higher the price the lower the cutoff productivity — a larger fraction of firms will find it profitable to operate when the price is high.

(b) Given the once-and-for-all choices, the value function for a firm is

$$v(z;p) = \max\left[0\,,\,\sum_{t=0}^{\infty}\beta^t\pi(z;p)\right] = \max\left[0,\,\frac{\pi(z;p)}{1-\beta}\right]$$

so that

$$v(z;p) = \frac{1}{1-\beta}\left[(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}\,(pz)^{\frac{1}{1-\alpha}} - k\right], \qquad z \geq z^*(p)$$

and $v(z;p) = 0$ for all $z < z^*(p)$.

(c) The free-entry condition can be written

$$k_e = \beta\int v(z;p)\,g(z)\,dz$$

where

$$\beta\int v(z;p)\,g(z)\,dz = \beta\int_{z^*}^{\infty}\frac{1}{1-\beta}\left[(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}\,(pz)^{\frac{1}{1-\alpha}} - k\right]g(z)\,dz$$

$$= \frac{\beta k}{1-\beta}\int_{z^*}^{\infty}\left[\frac{(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}\,(pz)^{\frac{1}{1-\alpha}}}{k} - 1\right]g(z)\,dz$$

$$= \frac{\beta k}{1-\beta}\int_{z^*}^{\infty}\left[\frac{(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}\,(pz)^{\frac{1}{1-\alpha}}}{(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}\,(pz^*)^{\frac{1}{1-\alpha}}} - 1\right]g(z)\,dz$$

$$= \frac{\beta k}{1-\beta}\int_{z^*}^{\infty}\left[\left(\frac{z}{z^*}\right)^{\frac{1}{1-\alpha}} - 1\right]g(z)\,dz$$

where the dependence on p has been substituted out using the zero-profit condition that defines the cutoff productivity. We can thus write the free-entry condition in terms of a single unknown, the cutoff z^*

$$k_e = \frac{\beta k}{1-\beta}\int_{z^*}^{\infty}\Big[\Big(\frac{z}{z^*}\Big)^{\frac{1}{1-\alpha}} - 1\Big]\,g(z)\,dz$$

Now define a composite parameter

$$\theta \equiv \frac{k_e}{k}\frac{1-\beta}{\beta} > 0$$

and the function

$$J(x) \equiv \int_{x}^{\infty}\Big[\Big(\frac{z}{x}\Big)^{\frac{1}{1-\alpha}} - 1\Big]\,g(z)\,dz$$

So in these terms the free-entry condition can be written

$$J(z^*) = \theta$$

which implicitly determines $z^*(\theta)$. By the implicit function theorem

$$J'(z^*(\theta))\frac{dz^*}{d\theta} = 1$$

so

$$\frac{dz^*}{d\theta} = \frac{1}{J'(z^*(\theta))}$$

Using Leibnitz's Rule, the derivative of $J(x)$ is

$$J'(x) = -\Big[\Big(\frac{x}{x}\Big)^{\frac{1}{1-\alpha}} - 1\Big](1) + \int_{x}^{\infty}\frac{\partial}{\partial x}\Big[\Big(\frac{z}{x}\Big)^{\frac{1}{1-\alpha}} - 1\Big]\,g(z)\,dz$$

$$= 0 + \int_{x}^{\infty}\frac{\partial}{\partial x}\Big(\frac{z}{x}\Big)^{\frac{1}{1-\alpha}}\,g(z)\,dz$$

$$= -\int_{x}^{\infty}\frac{1}{1-\alpha}\Big(\frac{z}{x}\Big)^{\frac{1}{1-\alpha}}\frac{1}{x}\,g(z)\,dz < 0$$

Since $J(x)$ is strictly decreasing in x we can conclude

$$\frac{dz^*}{d\theta} = \frac{1}{J'(z^*(\theta))} < 0$$

So any change that increases the composite parameter θ reduces the cutoff productivity z^*. In particular, an increase in k_e, a decrease in k, or a decrease in β all reduce the cutoff productivity. Once z^* has been obtained in this way we can recover p^* from the cutoff condition

$$p^* = \Big(\frac{k^{1-\alpha}}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}}\Big)\frac{1}{z^*}$$

So, for example, an increase in k_e or a decrease in β will increase p^*.

(d) With the Pareto density we can explicitly calculate the $J(x)$ function

$$J(x) \equiv \int_x^\infty \left[\left(\frac{z}{x}\right)^{\frac{1}{1-\alpha}} - 1 \right] g(z)\, dz$$

$$= \int_x^\infty \left[\left(\frac{z}{x}\right)^{\frac{1}{1-\alpha}} - 1 \right] \xi z^{-\xi-1}\, dz$$

$$= \int_x^\infty \left(\frac{z}{x}\right)^{\frac{1}{1-\alpha}} \xi z^{-\xi-1}\, dz - \int_x^\infty \xi z^{-\xi-1}\, dz$$

$$= x^{-\frac{1}{1-\alpha}} \left[-\frac{\xi}{\xi - \frac{1}{1-\alpha}} z^{-(\xi - \frac{1}{1-\alpha})} \Big|_{z=x}^{\infty} \right] - \left[-\frac{\xi}{\xi} z^{-\xi} \Big|_{z=x}^{\infty} \right]$$

$$= \frac{\xi}{\xi - \frac{1}{1-\alpha}} x^{-\xi} - x^{-\xi}$$

$$= \frac{\frac{1}{1-\alpha}}{\xi - \frac{1}{1-\alpha}} x^{-\xi}$$

where it is assumed that $\xi > \frac{1}{1-\alpha}$ so that the various integrals converge. Thus in the Pareto case we can write the free-entry condition as

$$\frac{\frac{1}{1-\alpha}}{\xi - \frac{1}{1-\alpha}} z^{*\,-\xi} = \frac{k_e}{k} \frac{1-\beta}{\beta}$$

which solves for

$$z^* = \left(\frac{\frac{1}{1-\alpha}}{\xi - \frac{1}{1-\alpha}} \frac{k}{k_e} \frac{\beta}{1-\beta} \right)^{1/\xi}$$

We then have

$$p^* = \left(\frac{k^{1-\alpha}}{\alpha^\alpha (1-\alpha)^{1-\alpha}} \right) \left(\frac{\frac{1}{1-\alpha}}{\xi - \frac{1}{1-\alpha}} \frac{k}{k_e} \frac{\beta}{1-\beta} \right)^{-1/\xi}$$

The ex post productivity distribution of actively producing firms is also Pareto with shape parameter ξ but with lower bound z^* rather than the lower bound 1 for the ex ante distribution $g(z)$.

2. **Hopenhayn with aggregate risk.** Firms discount flow profits according to a constant discount factor $0 < \beta < 1$. There is an unlimited number of potential entrants. On paying a sunk entry cost $k_e > 0$, an entrant receives an initial productivity draw $z_0 \sim g(z_0)$ and then starts operating the next period as an incumbent firm. On paying a fixed operating cost $k > 0$, an incumbent firm that hires n workers produces flow output $y = zn^\alpha$ with $0 < \alpha < 1$. The firm's productivity z evolves according to a Markov process with transition density $f(z' \,|\, z)$.

Unlike the basic Hopenhayn model, the demand curve facing the firms fluctuates. Let $D_t(p_t) = d_t/p_t$ denote the demand facing firms if the price is p_t and the state of demand is d_t. The state of demand d_t evolves according to a 2-state Markov chain $d_t \in \{d_l, d_h\}$ with transition probabilities $h(d' \mid d)$. Let $w_t = 1$ be the numeraire.

(a) What are the aggregate state variables in this economy? Setup the dynamic programming problem for incumbent firms and define a recursive competitive equilibrium for this economy. Be clear as to how all of the endogenous variables are determined in this equilibrium.

(b) Does the cross-sectional distribution of productivity fluctuate in this economy? Why or why not? What about the price p_t and the mass of entrants m_t? Explain.

(c) Outline an algorithm by which approximate solutions to this model can be computed.

SOLUTIONS:

(a)-(b) The individual state variable for an incumbent firm is its productivity z. The aggregate state variables are the state of demand d which evolves exogenously and the cross-sectional distribution of active producers $\mu(z)$ which evolves endogenously. Notice that even though the fluctuations in z are exogenous at the firm level, the distribution $\mu(z)$ will fluctuate endogenously because fluctuations in d will trigger fluctuations in the market-clearing price which will in turn trigger fluctuations in entry and exit.

Let the perceived law of motion for the distribution be

$$\mu' = H(\mu, d, d')$$

Let $p(d, \mu)$ denote the price in aggregate state d, μ and let $v(z\,;\,d, \mu)$ denote the value of incumbency to a firm with current productivity z. This value function solves the Bellman equation

$$v(z\,;\,d,\mu) = \pi(z, p(d,\mu)) + \beta \max\left[0\,,\, \sum_{d'} \int v(z'\,;\,d',\mu')\, f(z' \mid z) h(d' \mid d)\, dz'\right]$$

where

$$\pi(z,p) = (1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}(pz)^{\frac{1}{1-\alpha}} - k$$

subject to the perceived law of motion

$$\mu' = H(\mu, d, d')$$

The RHS of the Bellman equation implies an *exit threshold* $z^*(d,\mu)$ such that firm exits if $z < z^*(d,\mu)$. For interior cases, this cutoff is implicitly defined by

$$\sum_{d'} \int v(z'\,;\,d', H(\mu,d,d'))\, f(z' \mid z^*) h(d' \mid d)\, dz' = 0$$

A *stationary equilibrium* is a value function $v(z;d,\mu)$, output policy $y(z;d,\mu)$, cutoff productivity $z^*(d,\mu)$, law of motion for the distribution $H(\mu,d,d')$, mass of entrants $m(d,\mu)$, and pricing function $p(d,\mu)$ such that:

(i) taking $p(d,\mu)$ as given, $v(z;d,\mu)$, $y(z;d,\mu)$ and $z^*(d,\mu)$ solve the dynamic programming problem for an incumbent of type z

(ii) the free-entry condition

$$\beta \sum_{d'} \int v(z'; d', H(\mu, d, d'))\, g(z') h(d' \,|\, d)\, dz' \leq k_e$$

is satisfied, with strict equality whenever $m(d, \mu) > 0$

(iii) the goods market clears

$$\int y(z; d, \mu) \mu(z)\, dz = \frac{d}{p(d, \mu)}$$

(iv) the law of motion $H(\mu, d, d')$ is generated by the exit policy $z^*(d, \mu)$, the exogenous Markov chains $h(d' \,|\, d)$ and $f(z' \,|\, z)$ and the entry distribution $g_0(z)$

Note that entrants know neither their initial productivity state nor the aggregate state in the first period for which they operate, hence the free-entry condition takes expectations with respect to the joint distribution of both these outcomes.

(c) (Sketch) Approximate the distribution $\mu(z)$ with a finite vector of moments $\boldsymbol{m}$ and let

$$\boldsymbol{m}' = \hat{H}(\boldsymbol{m}, d, d')$$

denote the law of motion for this vector of moments. Let $\boldsymbol{a}_d, \boldsymbol{b}_d$ etc denote the coefficients of this approximate law of motion (conditional on the exogenous state d). Then start with an initial guess at coefficients for law of motion $\boldsymbol{a}_d^0, \boldsymbol{b}_d^0$ and solve the dynamic programming problem of an incumbent firm conditional on those coefficients. This also implies an exit policy. Find the price and mass of entrants that solve the free-entry condition and market clearing condition in the usual Hopenhayn way (conditional on these coefficients $\boldsymbol{a}_d^0, \boldsymbol{b}_d^0$). Then generate next period's distribution of producers using the exit policy, mass of entrants, and the exogenous Markov chains. Check if next period's distribution is approximately the same as predicted by the approximate law of motion with coefficients $\boldsymbol{a}_d^0, \boldsymbol{b}_d^0$. If not, update to new coefficients $\boldsymbol{a}_d^1, \boldsymbol{b}_d^1$ and try again.