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Macroeconomics
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Macroeconomics
Problem Set #3: Solutions

1. **Lognormal bond pricing.** Suppose the representative consumer has endowment y_t and can trade in a *riskless one-period bond* that pays 1 unit of consumption for sure one period after they are bought. Let q_t denote the price of the bond in period t and let a_t denote their holdings of bonds at the beginning of period t. The representative consumer seeks to maximize

$$\mathbb{E}\left\{\sum_{t=0}^{\infty} e^{-\rho t}\, u(c_t)\right\}, \qquad \rho > 0$$

subject to the budget constraints

$$c_t + q_t a_{t+1} = a_t + y_t$$

with initial conditions $a_0 = 0$ and $y_0 = 1$. Endowment growth $x_{t+1} \equiv y_{t+1}/y_t$ follows a Markov process with transition probabilities $F(x' \,|\, x) = \text{Prob}[x_{t+1} \leq x' \,|\, x_t = x]$.

(a) Let $q(x,y)$ denote the price of the bond in state x, y and let $v(a,x,y)$ denote the representative consumer's value function. Set up the representative consumer's dynamic programming problem in terms of this value function and define a recursive competitive equilibrium.

(b) Use the optimality conditions of the representative consumer and market clearing to solve for the equilibrium bond price $q(x,y)$.

Now suppose that the utility function has the CRRA form

$$u(c) = \frac{c^{1-\alpha} - 1}{1-\alpha}, \qquad \alpha > 0$$

and that the dividend growth process follows a lognormal AR(1) process

$$\log x_{t+1} = (1-\phi)g + \phi \log x_t + \varepsilon_{t+1}, \qquad -1 < \phi < 1, \qquad g \geq 0$$

where the innovations ε_t are IID $N(0, \sigma_\varepsilon^2)$.

(c) Show that the equilibrium bond price can be written $q(x)$ independent of the level y. Let $r(x) \equiv -\log q(x)$ denote the associated riskless interest rate. Solve for $q(x)$ and $r(x)$. Does higher α increase or decrease $r(x)$? Give as much intuition as you can.

Now suppose that there can be trade in bonds of longer maturity. Let a bond of maturity $j = 1, 2, \ldots$ pay 1 unit of consumption for sure sure in j periods time. Let q_t^j denote the price at t of a bond of maturity j. In this notation, the price of a one-period bond is $q_t^1 = q(x_t)$.

(d) Show that the price q_t^j of a bond of maturity j satisfies

$$q_t^j = \mathbb{E}_t\left[\, e^{-\rho}\, x_{t+1}^{-\alpha}\, q_{t+1}^{j-1}\,\right], \qquad j = 1, 2, \ldots$$

(with the convention that $q_t^0 = 1$). Solve for the equilibrium bond prices q_t^j.

(e) Let r_t^j denote the *yield* on a bond of maturity j

$$r_t^j \equiv -\frac{1}{j}\log q_t^j$$

In this notation, the one-period riskless rate is $r_t^1 = r(x_t)$. Solve for the equilibrium yields r_t^j. The *yield-curve* at date t is a plot of r_t^j against j. How does the yield curve depend on x_t? Is the yield curve in this economy upward or downward sloping in j? How if at all do your answers depend on ϕ? Explain.

SOLUTIONS:

(a) Taking $q(x, y)$ as given, the Bellman equation for the representative consumer can be written

$$v(a, x, y) = \max_{a'}\left[u(y + a - q(x, y)a') + e^{-\rho}\int v(a', x', x'y)\, dF(x' \,|\, x)\right]$$

where $y' = x'y$ has been used on the RHS. Let $a' = g(a, x, y)$ denote the optimal policy that achieves the maximum on the RHS of the Bellman equation. A *recursive competitive equilibrium* is three functions, $v(a, x, y)$, $g(a, x, y)$ and $q(x, y)$ such that (i) taking $q(x, y)$ as given $v(a, x, y)$ and $g(a, x, y)$ solve the representative consumer's dynamic programming problem, and (ii) markets clear, $g(a, x, y) = 0$ for all a, x, y (i.e., bonds are in zero net supply). If the bond market clears then so does the goods market, $c = y$.

(b) The first order condition for the representative consumer can be written

$$u_1(c)\, q(x, y) = e^{-\rho}\int v_1(a', x', x'y)\, dF(x' \,|\, x)$$

where it is understood that c is evaluated at the optimum. Applying the envelope theorem

$$v_1(a, x, y) = u_1(c)$$

Using this expression to eliminate $v_1(a', x', x'y)$ from the first order condition gives

$$u_1(c)\, q(x, y) = e^{-\rho}\int u_1(c')\, dF(x' \,|\, x)$$

Dividing both sides by $u_1(c)$ and using $c = y$ and $c' = y' = x'y$ then gives the equilibrium bond price

$$q(x, y) = e^{-\rho}\int \frac{u_1(x'y)}{u_1(y)}\, dF(x' \,|\, x)$$

(c) With the CRRA utility function $u_1(c) = c^{-\alpha}$ so our solution for the bond price becomes

$$q(x,y) = e^{-\rho} \int \frac{(x'y)^{-\alpha}}{y^{-\alpha}} \, dF(x' \,|\, x)$$

$$= e^{-\rho} \int x'^{-\alpha} \, dF(x' \,|\, x)$$

Since the RHS is independent of the current y we can write this more simply as

$$q(x) = e^{-\rho} \int x'^{-\alpha} \, dF(x' \,|\, x)$$

or

$$q(x) = e^{-\rho} \, \mathbb{E}[x'^{-\alpha} \,|\, x]$$

We now need to compute the conditional expectation on the RHS. To do this, recall that if a variable z is lognormally distributed $\log z \sim N(\mu_z, \sigma_z^2)$ then

$$\mathbb{E}[\exp(z)] = \exp(\mu_z + \sigma_z^2/2)$$

Now observe that

$$-\alpha \log x' \;\sim\; N(\, -\alpha((1-\phi)g + \phi \log x) \,,\; \alpha^2 \sigma_\varepsilon^2 \,)$$

so that

$$\mathbb{E}[x'^{-\alpha} \,|\, x] = \exp\big(-\alpha((1-\phi)g + \phi \log x) + \alpha^2 \sigma_\varepsilon^2/2\big)$$

Hence the solution for the bond price is

$$q(x) = \exp\big(-\rho - \alpha((1-\phi)g + \phi \log x) + \alpha^2 \sigma_\varepsilon^2/2\big)$$

with associated riskless interest rate

$$r(x) = -\log q(x) = \rho + \alpha((1-\phi)g + \phi \log x) - \alpha^2 \sigma_\varepsilon^2/2$$

The parameter α has two effects: (i) if the expected growth rate $(1-\phi)g+\phi\log x$ is positive, then a higher α increases the interest rate. This is the standard intertemporal substitution effect — the higher is α, the lower is the elasticity of intertemporal substitution $1/\alpha$ and so interest rates have to be higher to induce consumers to accept growing consumption. And (ii) a precautionary savings effect that drives the interest rate down. To see the former effect clearly, suppose growth is constant with $\log x = g$ for sure ($\sigma_\varepsilon = 0$). Then we would simply have

$$r = \rho + \alpha g$$

or in more familiar Euler equation terms

$$g = \log \frac{c_{t+1}}{c_t} = \frac{r - \rho}{\alpha}$$

(d) Consider two different ways to buy a riskless claim to one unit of consumption in two period's time. One way is to buy a $j = 2$ bond with price q_t^2 today. The other way is to *replicate* this by buying two one period bonds. Working backwards, we wait till $t+1$ and buy a $j = 1$ bond that will deliver the desired one unit of consumption and this has price q_{t+1}^1. But to execute this we need to have q_{t+1}^1 units of consumption at period $t+1$, that is at date t we need to buy a sure claim to q_{t+1}^1 units of consumption.

This can be evaluated using contingent claims. Let us consider an asset that *pays* $q^1(x')$ in period $t+1$. Such an asset will let us buy a $j = 1$ bond in state x' that will then deliver one unit of consumption for sure in $t+2$. With CRRA utility, the stochastic discount factor is $e^{-\rho}x'^{-\alpha}$ and an asset that pays $q^1(x')$ will have price

$$q^2(x) = e^{-\rho} \int x'^{-\alpha}\, q^1(x')\, dF(x' \,|\, x)$$

Repeating this argument for any two maturities j and $j-1$ we have

$$q^j(x) = e^{-\rho} \int x'^{-\alpha}\, q^{j-1}(x')\, dF(x' \,|\, x), \qquad j = 1, 2, \ldots$$

(notice that our result in part (c) is the special case $j = 1$ with the convention that $q^0 = 1$). In time series notation this is indeed

$$q_t^j = e^{-\rho}\, \mathbb{E}_t \left[x_{t+1}^{-\alpha}\, q_{t+1}^{j-1} \right], \qquad j = 1, 2, \ldots$$

where in this case conditioning on time-t information just means conditioning on x_t.

To solve for the bond prices, first write

$$q_t^1 = e^{-\rho}\, \mathbb{E}_t \left[x_{t+1}^{-\alpha} \right]$$

Hence

$$q_t^2 = e^{-\rho}\, \mathbb{E}_t \left[x_{t+1}^{-\alpha}\, e^{-\rho} \mathbb{E}_{t+1} \left[x_{t+2}^{-\alpha} \right]\right]$$

$$= e^{-\rho 2}\, \mathbb{E}_t \left[x_{t+1}^{-\alpha}\, \mathbb{E}_{t+1} \left[x_{t+2}^{-\alpha} \right]\right]$$

so by the law of iterated expectations

$$q_t^2 = e^{-\rho 2}\, \mathbb{E}_t \left[x_{t+1}^{-\alpha} x_{t+2}^{-\alpha} \right]$$

Similarly for any $j \geq 1$ we have

$$q_t^j = e^{-\rho j}\, \mathbb{E}_t \Big[\prod_{k=1}^{j} x_{t+k}^{-\alpha} \Big]$$

Now consider the random variable

$$s_t^j \equiv \prod_{k=1}^{j} x_{t+k}^{-\alpha}$$

Since this is the product of lognormal random variables it is also lognormal. We now need to figure out the moments of this random variable. First write

$$\log s_t^j = -\alpha \sum_{k=1}^{j} \log x_{t+k}$$

Then iterating forward from t to $t+k$ we have

$$\log x_{t+k} = g + \phi^k(\log x_t - g) + \sum_{i=1}^{k} \phi^{k-i}\varepsilon_{t+i}$$

Thus

$$\mathbb{E}_t[\log x_{t+k}] = g + \phi^k(\log x_t - g)$$

(i.e., expected growth mean reverts at rate ϕ to the long-run growth g) and hence

$$\mathbb{E}_t[\log s_t^j] = -\alpha \mathbb{E}_t\Big[\sum_{k=1}^{j} \log x_{t+k}\Big] = -\alpha \sum_{k=1}^{j} \mathbb{E}_t[\log x_{t+k}]$$

$$= -\alpha \sum_{k=1}^{j} [g + \phi^k(\log x_t - g)]$$

$$= -\alpha \left(jg + (\log x_t - g) \sum_{k=1}^{j} \phi^k \right)$$

$$= -\alpha \left(jg + \phi \frac{1-\phi^j}{1-\phi}(\log x_t - g) \right)$$

Similarly the variance terms are given by

$$\text{Var}_t[\log x_{t+k}] = \sum_{i=1}^{k} \phi^{2(k-i)}\sigma_\varepsilon^2 = \phi^{2k}\sigma_\varepsilon^2 \sum_{i=1}^{k} \phi^{-2i} = \phi^{2k}\sigma_\varepsilon^2 \phi^{-2} \frac{1-\phi^{-2k}}{1-\phi^{-2}} = \frac{1-\phi^{2k}}{1-\phi^2}\sigma_\varepsilon^2$$

Hence

$$\text{Var}_t[\log s_t^j] = \sum_{k=1}^{j} \text{Var}_t[-\alpha \log x_{t+k}]$$

$$= \alpha^2 \sum_{k=1}^{j} [\frac{1-\phi^{2k}}{1-\phi^2}\sigma_\varepsilon^2]$$

$$= \frac{\alpha^2}{1-\phi^2} \left(j - \sum_{k=1}^{j} \phi^{2k} \right) \sigma_\varepsilon^2$$

$$= \frac{\alpha^2}{1-\phi^2} \left(j - \phi^2 \frac{1-\phi^{2j}}{1-\phi^2} \right) \sigma_\varepsilon^2$$

Note that the conditional covariances between $\log x_{t+1}$ and any $\log x_{t+k}$ is zero, given that the innovations ε_{t+k} are IID. We can then conclude that

$$q_t^j = \exp(-\rho j)\exp(\mathbb{E}_t[\log s_t^j] + \mathrm{Var}_t[\log s_t^j]/2)$$

$$= \exp\left(-\rho j - \alpha\left(jg + \phi\frac{1-\phi^j}{1-\phi}(\log x_t - g)\right) + \frac{\alpha^2}{1-\phi^2}\left(j - \phi^2\frac{1-\phi^{2j}}{1-\phi^2}\right)\frac{\sigma_\varepsilon^2}{2}\right)$$

(e) Hence the yields are

$$r_t^j \equiv -\frac{1}{j}\log q_t^j = \rho + \alpha\left(g + \frac{\phi}{j}\frac{1-\phi^j}{1-\phi}(\log x_t - g)\right) - \frac{\alpha^2}{1-\phi^2}\left(1 - \frac{\phi^2}{j}\frac{1-\phi^{2j}}{1-\phi^2}\right)\frac{\sigma_\varepsilon^2}{2}$$

As a quick sanity check on all this algebra, note that for $j = 1$ we get

$$r_t^1 = \rho + \alpha\left(g + \frac{\phi}{1}\frac{1-\phi^1}{1-\phi}(\log x_t - g)\right) - \frac{\alpha^2}{1-\phi^2}\left(1 - \frac{\phi^2}{1}\frac{1-\phi^2}{1-\phi^2}\right)\frac{\sigma_\varepsilon^2}{2}$$

$$= \rho + \alpha\left(g + \phi(\log x_t - g)\right) - \alpha^2\frac{\sigma_\varepsilon^2}{2}$$

which coincides with our answer from part (c) above.

In short, the yields have a 'one-factor' structure of the form

$$r_t^j = \bar{r} + a(j) + b(j)\log x_t$$

where the 'one-factor' that drives all yields is the realized growth rate $\log x_t$ and where the coefficients are given by

$$\bar{r} \equiv \rho + \alpha g - \frac{\alpha^2}{1-\phi^2}\frac{\sigma_\varepsilon^2}{2}$$

$$a(j) \equiv -\alpha g\frac{\phi}{j}\frac{1-\phi^j}{1-\phi}$$

$$b(j) \equiv \alpha\frac{\phi}{j}\frac{1-\phi^j}{1-\phi}$$

The responses of the yields r_t^j to the current growth rate $\log x_t$ are given by the slope coefficients $b(j)$. The slope coefficients $b(j)$ have the same sign as ϕ (since $|\phi| < 1$), so if growth is positively serially correlated then all the yields increase together but if growth is negatively serially correlated then all the yields decrease together. If growth is IID, $\phi = 0$, the yield curve is flat and independent of $\log x_t$ [as in the midsemester exam].

Irrespective of the sign, the magnitude of the effect is diminishing in the maturity j, i.e., short-rates are more sensitive to the current state of the economy than are long-rates. For $j \to \infty$ we have

$$\lim_{j\to\infty} r_t^j = \rho + \alpha g - \frac{\alpha^2}{1-\phi^2}\frac{\sigma_\varepsilon^2}{2} \equiv \bar{r}$$

which is independent of the state.

2. **Risk-averse job search and savings.** Consider an unemployed worker with preferences

$$\mathbb{E}\left\{\sum_{t=0}^{\infty}\beta^t\, u(c_t)\right\}, \qquad 0<\beta<1$$

where $u(c_t)$ is strictly increasing and concave. Each period the worker draws an IID wage offer w from a distribution $F(w) = \text{Prob}[w_t \leq w]$. If they accept the wage offer they become employed and have $c_t = w$ until they lose their job. If they reject the wage offer they remain unemployed, consume benefits $c_t = b$ this period, and draw a new wage offer w' next period.

At the beginning of each period an employed worker loses their job with probability $\delta \in (0,1)$ and keeps their job with probability $1-\delta$. If a worker loses their job at the beginning of period t they spend period t unemployed, obtain benefits b, and then draw a new wage offer w' at the beginning of period $t+1$ (which they can then accept or reject, as usual). If the worker keeps their job at the beginning of period t their wage remains unchanged, i.e., the same as the wage they accepted when first starting their job.

(a) Let $v(w)$ denote the unemployed worker's value function. Setup and explain the unemployed worker's dynamic programming problem in terms of this value function.

(b) Show that the unemployed worker's problem is characterized by a reservation wage $\bar{w}$ such that the worker rejects the offer if $w < \bar{w}$ and accepts the offer if $w > \bar{w}$. How does $\bar{w}$ depend on δ? Explain.

Now suppose that workers can save. Let $n_t \in \{0,1\}$ denote a worker's beginning of period employment status, with $n_t = 1$ denoting employment and $n_t = 0$ denoting unemployment. The worker's income is then $y_t = w_t n_t + b(1-n_t)$. Suppose also that workers have beginning of period assets a_t and have budget constraints

$$c_t + a_{t+1} = Ra_t + y_t$$

for some constant return R and given initial condition a_0.

(c) Let $V(a,w,n)$ denote the value function of a worker with current assets a, wage (offer) w and who is in employment status $n \in \{0,1\}$. Setup and explain the worker's dynamic programming problem.

Now suppose that the utility function has the CRRA form

$$u(c) = \frac{c^{1-\alpha}-1}{1-\alpha}, \qquad \alpha > 0$$

and that the wage distribution is lognormal, i.e., that $\log w$ is IID $N(\mu_w, \sigma_w^2)$.

(d) Let the parameters be $\alpha = 1$, $\beta = 0.95$, $R = 1/\beta$, $\delta = 0.05$, $b = 0.4$, $\mu_w = -0.125$, and $\sigma_w = 0.5$. Using these parameter values, solve the worker's dynamic programming problem.

(e) Let $\bar{w}(a)$ denote the worker's reservation wage. How does the worker's reservation wage depend on their savings a? Explain.

(f) How would your answers to (d) and (e) change if instead $\sigma_w = 0.25$? or $\sigma_w = 1$? How would your answers to (d) and (e) change if instead $\delta = 0.025$? or $\delta = 0.1$? Explain.

SOLUTIONS:

(a) With the exogenous probability $\delta \in (0,1)$ of job loss an unemployed worker's Bellman equation is

$$v(w) = \max_{\text{accept, reject}} \left\{ \begin{array}{c} u(w) + \beta\Big[(1-\delta)v(w) + \delta\Big(u(b) + \beta \displaystyle\int_0^\infty v(w')\,dF(w')\Big)\Big] \\ \\ u(b) + \beta \displaystyle\int_0^\infty v(w')\,dF(w') \end{array} \right\}$$

To understand the first branch, observe that if the worker accepts the wage w they get current payoff $u(w)$ and then with probability $1-\delta$ they keep their job next period giving continuation value $v(w)$ but with probability δ they lose their job next period and go into at least a one-period spell of unemployment with flow payoff $u(b)$ before being able to sample their next w' the following period.

As usual, the Bellman equation characterizes the value $v(w)$ of being endowed with wage draw w at the beginning of the period and then proceeding optimally.

(b) Rejecting a wage offer gives an expected payoff

$$u(b) + \beta\bar{v}, \qquad \bar{v} \equiv \int_0^\infty v(w')\,dF(w')$$

that is independent of the current w. Accepting a wage offer gives an expected payoff

$$u(w) + \beta(1-\delta)v(w) + \beta\delta\big[u(b) + \beta\bar{v}\big]$$

that is strictly increasing in the current w [since by the envelope theorem $v(w)$ is nondecreasing in w]. If w is such that accepting is optimal, we have

$$v(w) = u(w) + \beta(1-\delta)v(w) + \beta\delta\big[u(b) + \beta\bar{v}\big], \qquad w : \text{accepting optimal}$$

Hence for these w we have

$$v(w) = \frac{1}{1-\beta(1-\delta)}\big\{u(w) + \beta\delta\big[u(b) + \beta\bar{v}\big]\big\}$$

In other words, there is a constant $\bar{w}$ such that it is optimal to accept all $w > \bar{w}$ and to reject all $w < \bar{w}$ where $\bar{w}$ satisfies the indifference condition

$$\frac{1}{1-\beta(1-\delta)}\big\{u(\bar{w}) + \beta\delta\big[u(b) + \beta\bar{v}\big]\big\} = u(b) + \beta\bar{v}$$

Cancelling common terms gives

$$\frac{1}{1-\beta}u(\bar{w}) = u(b) + \beta\bar{v}$$

Notice that the probability of job loss δ only influences the reservation wage $\bar{w}$ though $\bar{v}$. In terms of $\bar{w}$ the value function $v(w)$ has the piecewise form

$$v(w) = \begin{cases} \dfrac{1}{1-\beta}u(\bar{w}) & w \leq \bar{w} \\ \dfrac{1}{1-\beta(1-\delta)}\left\{u(w) + \beta\delta\big[u(b) + \beta\bar{v}\big]\right\} & w \geq \bar{w} \end{cases}$$

And using the indifference condition we can write

$$v(w) = \begin{cases} \dfrac{1}{1-\beta}u(\bar{w}) & w \leq \bar{w} \\ \dfrac{1}{1-\beta(1-\delta)}\left\{u(w) + \dfrac{\beta\delta}{1-\beta}u(\bar{w})\right\} & w \geq \bar{w} \end{cases}$$

Hence in terms of $\bar{w}$ the expected value $\bar{v}$ is

$$\begin{aligned} \bar{v} &\equiv \int_0^\infty v(w')\,dF(w') \\ &= \frac{1}{1-\beta}\int_0^{\bar{w}} u(\bar{w})\,dF(w') + \frac{1}{1-\beta(1-\delta)}\int_{\bar{w}}^\infty \left\{u(w') + \frac{\beta\delta}{1-\beta}u(\bar{w})\right\}dF(w') \end{aligned}$$

Now write the indifference condition as

$$\begin{aligned} &u(\bar{w}) + \frac{\beta}{1-\beta}u(\bar{w}) \\ &= u(b) + \frac{\beta}{1-\beta}\int_0^{\bar{w}} u(\bar{w})\,dF(w') + \frac{\beta}{1-\beta(1-\delta)}\int_{\bar{w}}^\infty \left\{u(w') + \frac{\beta\delta}{1-\beta}u(\bar{w})\right\}dF(w') \end{aligned}$$

Cancelling common terms then gives

$$u(\bar{w}) - u(b) = \frac{\beta}{1-\beta(1-\delta)}\left(\int_{\bar{w}}^\infty (u(w') - u(\bar{w}))\,dF(w')\right)$$

We can view this as one equation in one unknown, $\bar{w}$, and the solution $\bar{w}(\delta)$, say, depends on δ only because the RHS does. That is, the LHS $u(\bar{w}) - u(b)$ curve is independent of δ but the RHS curve is everywhere lower (compared to the usual model with $\delta = 0$). Intuitively, in the presence of the risk of job loss the expected benefit of searching again is lower (because the expected duration of any accepted wage is lower). There is a unique $\bar{w}(\delta)$ that solves this condition and since the RHS is decreasing in δ for each $\bar{w}$ the solution $\bar{w}(\delta)$ is decreasing in δ.

(c) **Preliminaries.** Let $V(a, w, n)$ denote the value function of a worker with current assets a, wage (offer) w, and who is in current employment status $n \in \{0, 1\}$. In particular, let $V(a, w, 0)$ denote the value function of an unemployed worker with wage offer w and let $V(a, w, 1)$ denote the value function of an employed worker with actual wage w. Finally let $U(a)$ denote value of declining a wage offer and searching again. These are related by the following system of Bellman equations. First, for an unemployed worker we have

$$V(a, w, 0) = \max_{n' \in \{0,1\}} \left[\, n'V(a, w, 1) + (1-n')U(a)\,\right]$$

so that if the unemployed worker sets $n' = 1$ they accept the wage offer w and become employed obtaining the value $V(a, w, 1)$ while if they set $n' = 0$ they reject the wage offer w and remain unemployed. Hence $V(a, w, 0)$ is the ex ante value with offer w in hand while $U(a)$ is the ex post value conditional on declining the wage offer. For an employed worker we have

$$V(a, w, 1) = \max_{a'} \left[u(w + Ra - a') + \beta \left((1 - \delta) V(a', w, 1) + \delta U(a') \right) \right]$$

so that they choose their asset holdings a' understanding that with probability $1 - \delta$ they keep their job and continue to work at wage w while with probability δ they lose their job and become unemployed. Notice that in keeping with parts (a)-(b) above if they lose their job they will go into at least a one-period spell of unemployment before being able to sample a new wage. Finally for an unemployed worker without a wage offer we have

$$U(a) = \max_{a'} \left[u(b + Ra - a') + \beta \int V(a', w', 0)\, dF(w') \right]$$

so that they get immediate benefits b and sample a wage w' next period.

Simplifying the problem using the reservation policy. An unemployed worker will accept all w such that $V(a, w, 1) > U(a)$ and reject all w such that $V(a, w, 1) < U(a)$. From the envelope condition we see that $V(a, w, 1)$ is strictly increasing in w so there is a reservation wage, $\bar{w}$ such that $V(a, w, 1) > U(a)$ for all $w > \bar{w}$ and $V(a, w, 1) < U(a)$ for all $w < \bar{w}$. The reservation wage is then characterized by the indifference condition

$$V(a, \bar{w}, 1) = U(a)$$

and hence, in general, is a function of the worker's asset level, $\bar{w}(a)$ say. To simplify the problem, let us define $\bar{v}(a) \equiv V(a, \bar{w}(a), 1)$ which of course simply equals $U(a)$. And in keeping with this notation we can define $v(a, w) \equiv V(a, w, 1)$. We can then write the worker's problem in terms of a *system* of these two value functions, $v(a, w)$ and $\bar{v}(a)$. In particular, we have

$$v(a, w) = \max_{a'} \left[u(w + Ra - a') + \beta \left((1 - \delta) v(a', w) + \delta \bar{v}(a') \right) \right]$$

and

$$\bar{v}(a) = \max_{a'} \left[u(b + Ra - a') + \beta \int \max \left\{ v(a', w') \, , \, \bar{v}(a') \right\} dF(w') \right]$$

The reservation wage $\bar{w}(a)$ is then implicitly determined by the indifference condition $v(a, \bar{w}) = \bar{v}(a)$.

(d) The attached Matlab code `ps3_question2.m` solves the model with the given parameters using collocation. One trick I used is to get good initial conditions for the problem I first solved the simpler problem where $\delta = 0$ to get the value function for a worker that never loses their job, i.e., I solved

$$\tilde{v}(a, w) = \max_{a'} \left[u(w + Ra - a') + \beta \tilde{v}(a', w) \right]$$

I then used this value function $\tilde{v}(a, w)$ to provide initial coefficients from which to iterate on the system in the two unknown functions $v(a, w)$ and $\bar{v}(a)$. Figure 1 below shows the values $v(a, w)$ as a function of a for each w. For each w, the values $v(a, w)$ are strictly increasing and strictly concave in a. Notice also that for each a the values $v(a, w)$ are strictly increasing in w. The sensitivity of $v(a, w)$ to a is greatest for low w. For high a, the value functions are much less sensitive to w — since for high a the workers are fairly well-insured against the idiosyncratic risk of job loss (at rate δ) + wage resampling. Figure 2 shows the associated consumption policy functions $c(a, w)$ which are likewise increasing in a and w.

(e) As discussed above, the reservation wage $\bar{w}(a)$ is implicitly determined by the indifference condition $v(a, \bar{w}) = \bar{v}(a)$. The green line in Figure 3 shows the value $v(a, w)$ as a function of the wage offer w for given a. The red line shows the associated $\bar{v}(a)$ which is independent of w. The dashed lines show the counterpart values for a higher level of assets. The worker rejects all offers $w < \bar{w}(a)$ and accepts all $w > \bar{w}(a)$. Figure 4 below plots $\bar{v}(a)$ against the family of $v(a, w)$ curves. Notice that at each a we have that $\bar{v}(a)$ is more steeply increasing in a than is $v(a, w)$, for any w. From the implicit function theorem we can say

$$\frac{\partial v}{\partial a} + \frac{\partial v}{\partial w}\frac{d\bar{w}}{da} = \frac{d\bar{v}}{da}$$

hence

$$\frac{d\bar{w}}{da} = \frac{\frac{d\bar{v}}{da} - \frac{\partial v}{\partial a}}{\frac{\partial v}{\partial w}} > 0$$

since as we have seen $\frac{d\bar{v}}{da} > \frac{\partial v}{\partial a} > 0$ and $\frac{\partial v}{\partial w} > 0$. That is, individuals with low asset levels will have lower reservation wages than individuals with high asset levels. The marginal value of assets in hand is higher for a worker who is unemployed than for an otherwise equivalent worker who is employed.

(f) You can obtain the answers for this part by rerunning `ps3_question2.m` as needed.

Increasing σ_w increases the amount of dispersion in wage offers. Since workers can always turn down wage offers, this presents workers with the possibility of more 'good' wage offers which tends to increase their reservation wage. But this higher dispersion also increases the incentives to save because it increases the likelihood of a large change in income conditional on job loss (this effect is absent in a model without saving).

Increasing δ to $\delta = 0.1$ increases the amount of risk facing workers, which increases their savings (for precautionary reasons, as in question 3 below) and reduces their reservation wage (as in parts (a) and (b) above). Similarly decreasing δ to $\delta = 0.025$ decreases the amount of risk facing workers, which decreases their savings and increases their reservation wage.

3. **Precautionary savings by backwards induction.** Consider a *finite horizon* savings problem where the representative consumer seeks to maximize

$$\mathbb{E}\left\{\sum_{t=0}^{T} \beta^t u(c_t)\right\}, \qquad 0 < \beta < 1$$

where $u(c_t)$ is strictly increasing and concave. Each period the consumer draws IID income y_t from a distribution $F(y) = \text{Prob}[y_t \leq y]$ and has budget constraints

$$c_t + a_{t+1} = Ra_t + y_t$$

for some constant return R and given initial condition a_0.

(a) Let $x \equiv Ra + y$ denote the consumer's beginning of period '*cash-on-hand*' and let $v_t(x)$ denote the time t value of having cash-on-hand x. Setup and explain the consumer's dynamic programming problem.

Again suppose that the utility function has the CRRA form

$$u(c) = \frac{c^{1-\alpha} - 1}{1 - \alpha}, \qquad \alpha > 0$$

(b) Show that the terminal value function $v_T(x)$ is strictly increasing, strictly concave, and exhibits *prudence*. Show that $v_{T-1}(x)$ has the same properties. Show by induction that the sequence of value functions $v_t(x)$ for $t = 0, 1, \ldots, T$ all have these properties.

Now suppose that the income distribution is lognormal, i.e., that $\log y$ is IID $N(\mu_y, \sigma_y^2)$.

(c) Let the parameters be $\alpha = 1$, $\beta = 0.95$, $R = 1/\beta$, $\mu_y = -0.125$, and $\sigma_y = 0.5$ and let the horizon be $T = 75$. Using these parameter values, solve the consumer's dynamic programming problem by backwards induction. Plot the consumer's value functions $v_t(x)$ and consumption policy functions $c_t(x)$.

(d) How would your answers to (c) change if instead $\alpha = 0.5$? or $\alpha = 2$? How would your answers to (c) change if instead $T = 50$? or $T = 100$? Explain.

SOLUTIONS:

(a) Defining cash-on-hand by $x \equiv Ra + y$ the consumer's budget constraint becomes $c + a' = x$. And since $x' = Ra' + y'$ we can also write the law of motion for cash on hand as $x' = R(x - c) + y'$. The time-t Bellman equation can then be written

$$v_t(x) = \max_{c \geq 0} \left[u(c) + \beta \int v_{t+1}(R(x - c) + y') dF(y') \right], \qquad t = 0, 1, ..., T$$

where $x' = Ra' + y'$ has been used on the RHS. This Bellman equation characterizes the value $v_t(x)$ of having cash-on-hand x at the beginning of period t and then proceeding optimally.

(b) For the terminal period $t = T$ it is optimal to choose $a_{T+1} = 0$ so that $c_T = x$ and hence $v_T(x) = u(x)$. Hence $v_T(x)$ inherits all of the properties of the utility function. In particular $v_T'(x) = x^{-\alpha} > 0$ and $v_T''(x) = -\alpha x^{-\alpha-1} < 0$ so the terminal value function is strictly increasing and strictly concave. The terminal value function exhibits *prudence* if the marginal value $v_T'(x)$ is convex, that is, if $v_T'''(x) \geq 0$. Since $v_T'''(x) = +\alpha(\alpha + 1)x^{-\alpha-2} > 0$, the terminal value function indeed also exhibits prudence.

At period $T - 1$ we then have

$$v_{T-1}(x) = \max_{c \geq 0} \left[u(c) + \beta \int u(R(x - c) + y') dF(y') \right]$$

where $v_T(x) = u(x)$ has been used on the RHS. By the envelope theorem

$$v_{T-1}'(x) = \beta R \int u'(R(x - c) + y') dF(y') > 0$$

where it is understood that c is evaluated at the optimum. The sign follows because $u'(R(x-c)+y')>0$ for every y' and hence the integral is positive. Similarly

$$v''_{T-1}(x) = \beta R^2 \int u''(R(x-c)+y')dF(y') < 0$$

and

$$v'''_{T-1}(x) = \beta R^3 \int u'''(R(x-c)+y')dF(y') > 0$$

At period $T-2$ we then have

$$v_{T-2}(x) = \max_{c\geq 0} \left[u(c) + \beta \int v_{T-1}(R(x-c)+y')dF(y') \right]$$

Again by the envelope theorem

$$v'_{T-2}(x) = \beta R \int v'_{T-1}(R(x-c)+y')dF(y') > 0$$

$$v''_{T-2}(x) = \beta R^2 \int v''_{T-1}(R(x-c)+y')dF(y') < 0$$

and

$$v'''_{T-2}(x) = \beta R^3 \int v'''_{T-1}(R(x-c)+y')dF(y') > 0$$

Proceeding in this way we see that the sequence of value functions $v_t(x)$ for $t = 0, 1, ..., T$ indeed inherit all of the properties of $u(x)$.

(c) The attached Matlab code `ps3_question3.m` solves the model with the given parameters using collocation iterating backwards from the terminal condition. Figure 4 below shows the value functions $v_t(x)$ for various t. Figure 5 below shows the associated consumption policy functions $c_t(x)$ that obtain the maximum on the RHS of the respective Bellman equations. The value functions $v_t(x)$ are indeed all strictly increasing and strictly concave. The value functions gradually descend and become generally flatter as t progresses. The speed at which the value functions descend picks up as t gets closer to T. The consumption functions $c_t(x)$ are all bounded above by x with $c_T(x) = x$ (the 45-degree line) in the last period. Notice that the consumption functions are lower and flatter for earlier t, i.e., the consumer is saving more, $a' = x - c_t(x)$ is larger, with $a' = x - c_t(x) \to 0$ as $t \to T$.

(d) You can obtain the answers for this part by rerunning `ps3_question3.m` as needed. Reducing α to $\alpha = 0.5$ reduces the concavity of $v_t(x)$ as $x \to 0$. Increasing α to $\alpha = 2$ increases the concavity of $v_t(x)$ as $x \to 0$, especially for high t. Changing the horizon T simply accelerates or decelerates the patterns documented in part (c).

Figure 1: Risk-averse job search: value functions

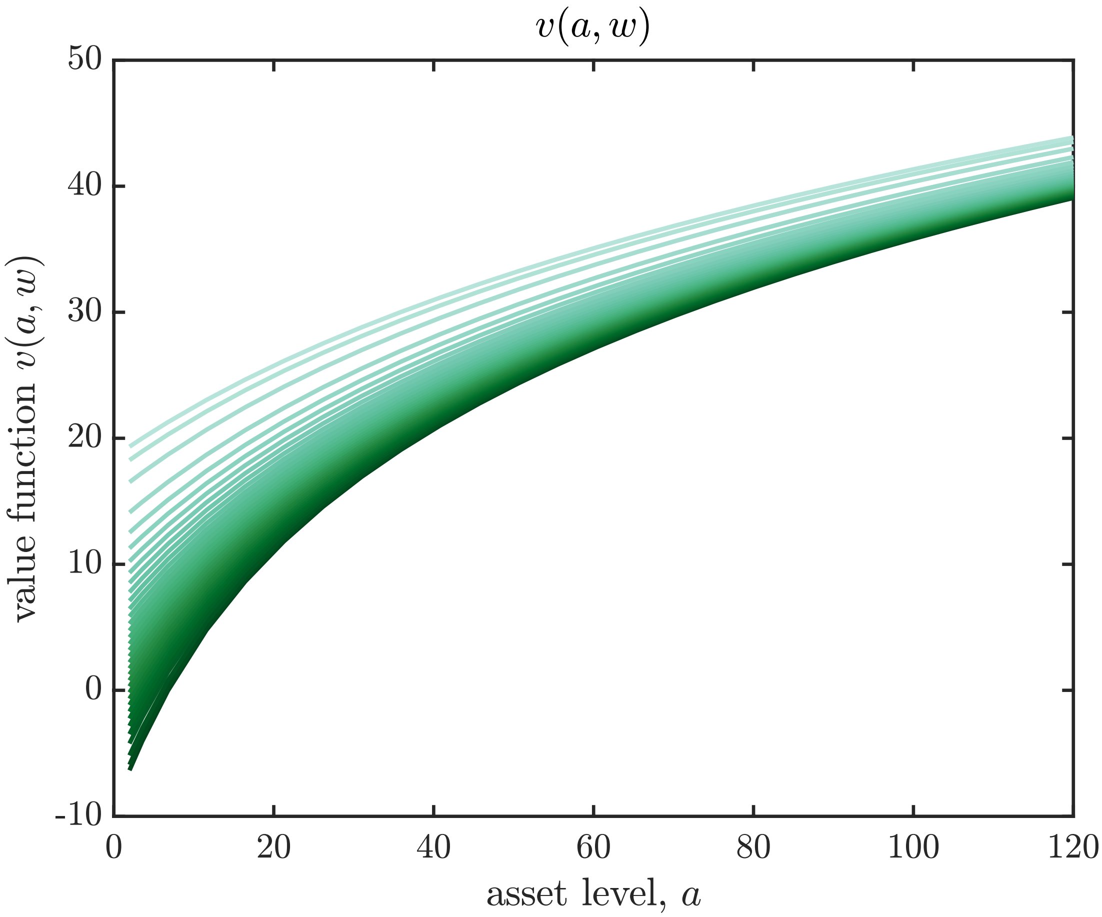

Figure 2: Risk-averse job search: consumption functions

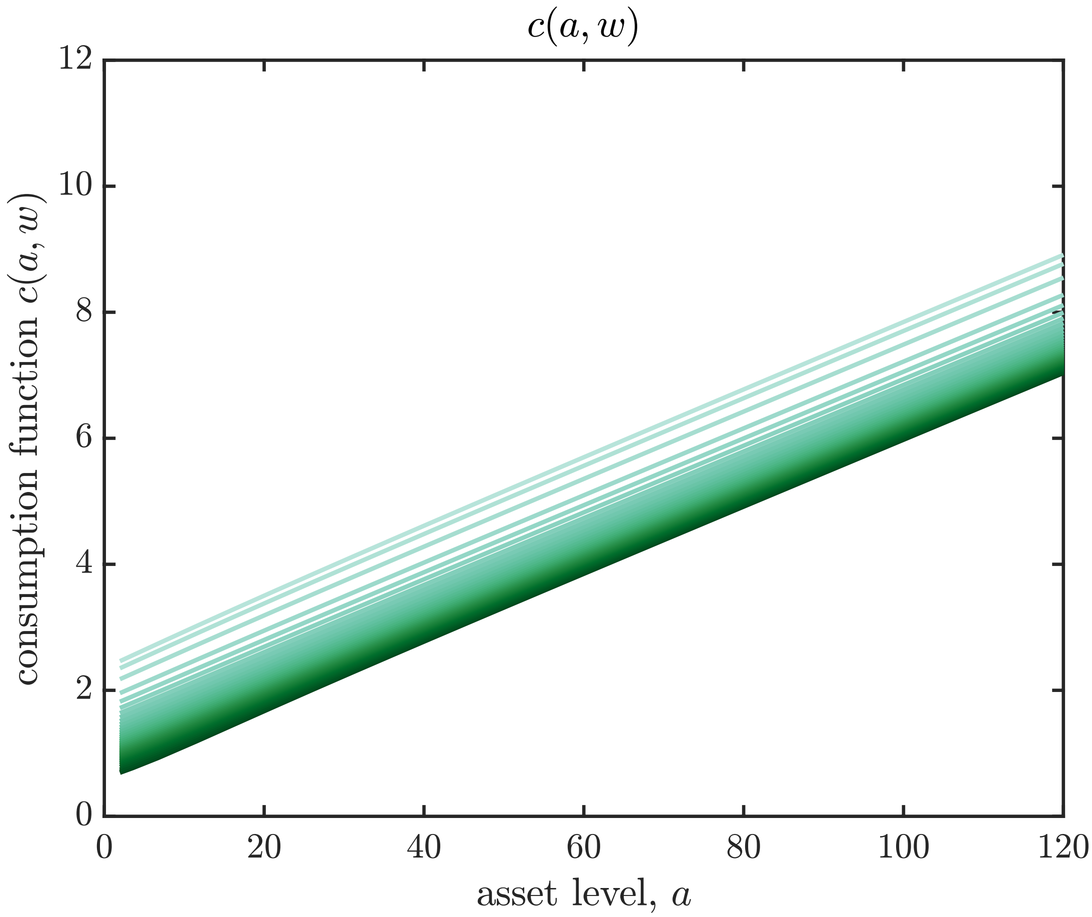

Figure 3: Determining the reservation wage $\bar{w}(a)$

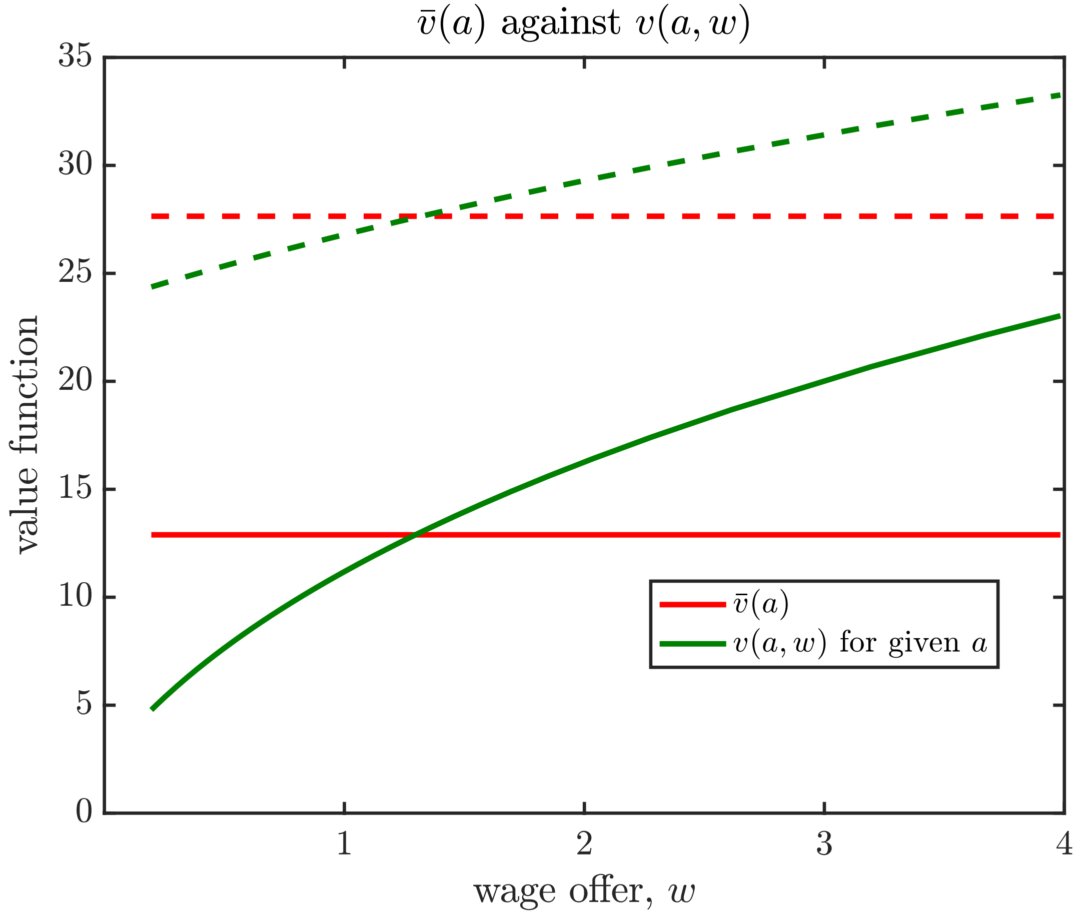

Figure 4: Marginal value of assets in hand higher for unemployed

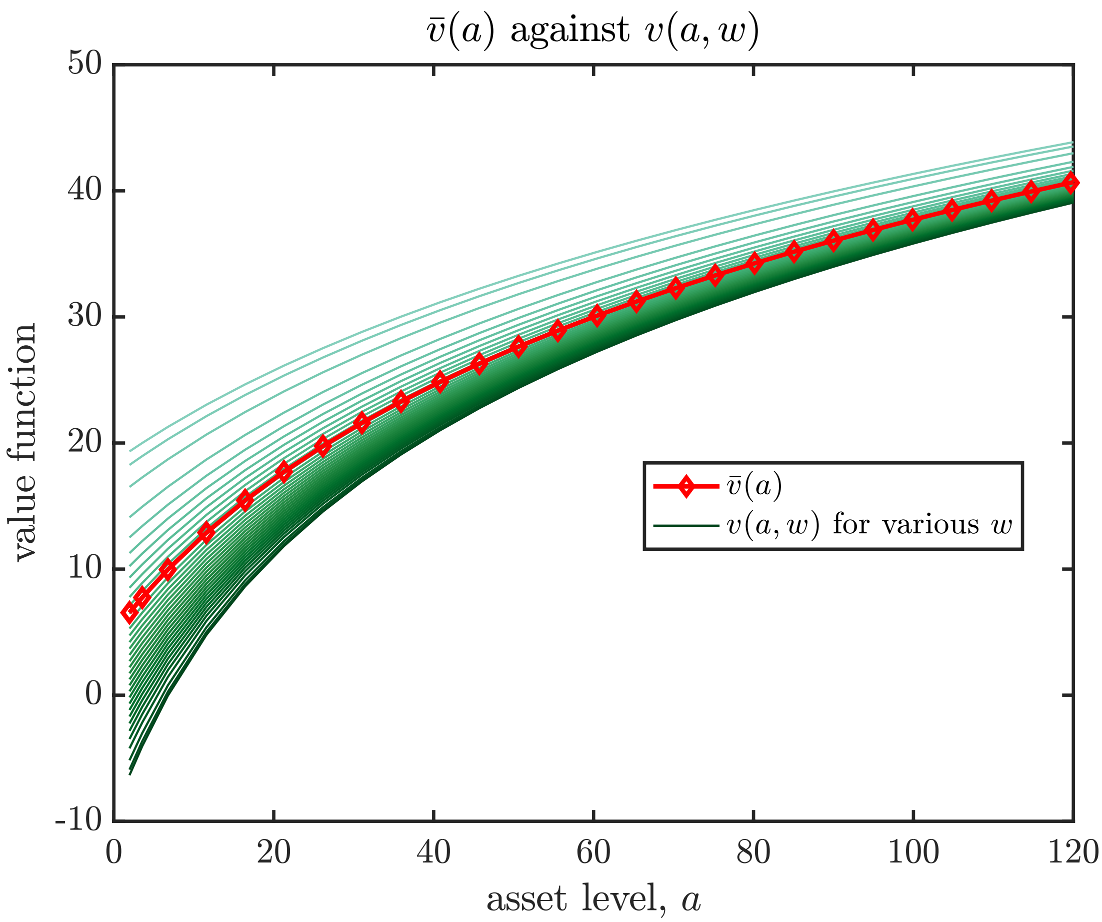

Figure 5: Finite-horizon model: value functions

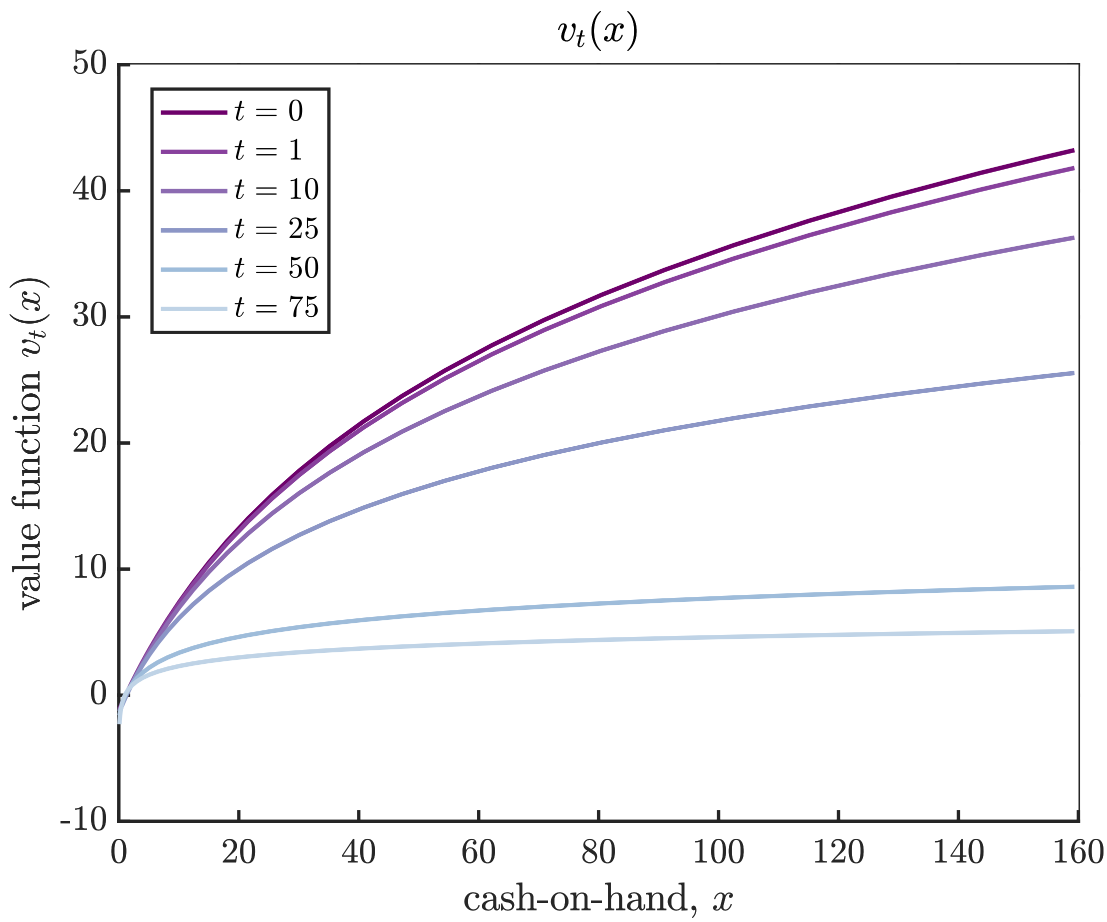

Figure 6: Finite-horizon model: value functions