Macroeconomics
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Macroeconomics: Problem Set #3

Due Tuesday May 14 in class

1. **Lognormal bond pricing.** Suppose the representative consumer has endowment y_t and can trade in a *riskless one-period bond* that pays 1 unit of consumption for sure one period after they are bought. Let q_t denote the price of the bond in period t and let a_t denote their holdings of bonds at the beginning of period t. The representative consumer seeks to maximize

$$\mathbb{E}\left\{\sum_{t=0}^{\infty} e^{-\rho t}\, u(c_t)\right\}, \qquad \rho > 0$$

subject to the budget constraints

$$c_t + q_t a_{t+1} = a_t + y_t$$

with initial conditions $a_0 = 0$ and $y_0 = 1$. Endowment growth $x_{t+1} \equiv y_{t+1}/y_t$ follows a Markov process with transition probabilities $F(x' \,|\, x) = \text{Prob}[x_{t+1} \leq x' \,|\, x_t = x]$.

   (a) Let $q(x, y)$ denote the price of the bond in state x, y and let $v(a, x, y)$ denote the representative consumer's value function. Set up the representative consumer's dynamic programming problem in terms of this value function and define a recursive competitive equilibrium.

   (b) Use the optimality conditions of the representative consumer and market clearing to solve for the equilibrium bond price $q(x, y)$.

Now suppose that the utility function has the CRRA form

$$u(c) = \frac{c^{1-\alpha} - 1}{1 - \alpha}, \qquad \alpha > 0$$

and that the dividend growth process follows a lognormal AR(1) process

$$\log x_{t+1} = (1 - \phi)g + \phi \log x_{t+1} + \varepsilon_{t+1}, \qquad -1 < \phi < 1, \qquad g \geq 0$$

where the innovations ε_t are IID $N(0, \sigma_\varepsilon^2)$.

   (c) Show that the equilibrium bond price can be written $q(x)$ independent of the level y. Let $r(x) \equiv -\log q(x)$ denote the associated riskless interest rate. Solve for $q(x)$ and $r(x)$. Does higher α increase or decrease $r(x)$? Give as much intuition as you can.

Now suppose that there can be trade in bonds of longer maturity. Let a bond of maturity $j = 1, 2, \ldots$ pay 1 unit of consumption for sure sure in j periods time. Let q_t^j denote the price at t of a bond of maturity j. In this notation, the price of a one-period bond is $q_t^1 = q(x_t)$.

(d) Show that the price q_t^j of a bond of maturity j satisfies

$$q_t^j = \mathbb{E}_t\left[\, e^{-\rho}\, x_{t+1}^{-\alpha}\, q_{t+1}^{j-1}\,\right], \qquad j = 1, 2, \ldots$$

(with the convention that $q_t^0 = 1$). Solve for the equilibrium bond prices q_t^j.

(e) Let r_t^j denote the *yield* on a bond of maturity j

$$r_t^j \equiv -\frac{1}{j} \log q_t^j$$

In this notation, the one-period riskless rate is $r_t^1 = r(x_t)$. Solve for the equilibrium yields r_t^j. The *yield-curve* at date t is a plot of r_t^j against j. How does the yield curve depend on x_t? Is the yield curve in this economy upward or downward sloping in j? How if at all do your answers depend on ϕ? Explain.

2. **Risk-averse job search and savings.** Consider an unemployed worker with preferences

$$\mathbb{E}\left\{ \sum_{t=0}^{\infty} \beta^t\, u(c_t) \right\}, \qquad 0 < \beta < 1$$

where $u(c_t)$ is strictly increasing and concave. Each period the worker draws an IID wage offer w from a distribution $F(w) = \text{Prob}[w_t \leq w]$. If they accept the wage offer they become employed and have $c_t = w$ until they lose their job. If they reject the wage offer they remain unemployed, consume benefits $c_t = b$ this period, and draw a new wage offer w' next period.

At the beginning of each period an employed worker loses their job with probability $\delta \in (0,1)$ and keeps their job with probability $1-\delta$. If a worker loses their job at the beginning of period t they spend period t unemployed, obtain benefits b, and then draw a new wage offer w' at the beginning of period $t+1$ (which they can then accept or reject, as usual). If the worker keeps their job at the beginning of period t their wage remains unchanged, i.e., the same as the wage they accepted when first starting their job.

(a) Let $v(w)$ denote the unemployed worker's value function. Setup and explain the unemployed worker's dynamic programming problem in terms of this value function.

(b) Show that the unemployed worker's problem is characterized by a reservation wage $\bar{w}$ such that the worker rejects the offer if $w < \bar{w}$ and accepts the offer if $w > \bar{w}$. How does $\bar{w}$ depend on δ? Explain.

Now suppose that workers can save. Let $n_t \in \{0,1\}$ denote a worker's beginning of period employment status, with $n_t = 1$ denoting employment and $n_t = 0$ denoting unemployment. The worker's income is then $y_t = w_t n_t + b(1-n_t)$. Suppose also that workers have beginning of period assets a_t and have budget constraints

$$c_t + a_{t+1} = R a_t + y_t$$

for some constant return R and given initial condition a_0.

(c) $v(w,a,n)$ denote the value function of a worker with current wage (offer) w, assets a and who is in employment status $n \in \{0,1\}$. Setup and explain the worker's dynamic programming problem.

Now suppose that the utility function has the CRRA form

$$u(c) = \frac{c^{1-\alpha} - 1}{1 - \alpha}, \qquad \alpha > 0$$

and that the wage distribution is lognormal, i.e., that $\log w$ is IID $N(\mu_w, \sigma_w^2)$.

(d) Let the parameters be $\alpha = 1$, $\beta = 0.95$, $R = 1/\beta$, $\delta = 0.05$, $b = 0.4$, $\mu_w = -0.125$, and $\sigma_w = 0.5$. Using these parameter values, solve the worker's dynamic programming problem.

(e) Let $\bar{w}(a)$ denote the worker's reservation wage. How does the worker's reservation wage depend on their savings a? Explain.

(f) How would your answers to (d) and (e) change if instead $\sigma_w = 0.25$? or $\sigma_w = 1$? How would your answers to (d) and (e) change if instead $\delta = 0.025$? or $\delta = 0.1$? Explain.

3. **Precautionary savings by backwards induction.** Consider a *finite horizon* savings problem where the representative consumer seeks to maximize

$$\mathbb{E}\left\{\sum_{t=0}^{T} \beta^t\, u(c_t)\right\}, \qquad 0 < \beta < 1$$

where $u(c_t)$ is strictly increasing and concave. Each period the consumer draws IID income y_t from a distribution $F(y) = \text{Prob}[y_t \leq y]$ and has budget constraints

$$c_t + a_{t+1} = Ra_t + y_t$$

for some constant return R and given initial condition a_0.

(a) Let $x \equiv Ra + y$ denote the consumer's beginning of period '*cash-on-hand*' and let $v_t(x)$ denote the time t value of having cash-on-hand x. Setup and explain the consumer's dynamic programming problem.

Again suppose that the utility function has the CRRA form

$$u(c) = \frac{c^{1-\alpha} - 1}{1 - \alpha}, \qquad \alpha > 0$$

(b) Show that the terminal value function $v_T(x)$ is strictly increasing, strictly concave, and exhibits *prudence*. Show that $v_{T-1}(x)$ has the same properties. Show by induction that the sequence of value functions $v_t(x)$ for $t = 0, 1, \ldots, T$ all have these properties.

Now suppose that the income distribution is lognormal, i.e., that $\log y$ is IID $N(\mu_y, \sigma_y^2)$.

(c) Let the parameters be $\alpha = 1$, $\beta = 0.95$, $R = 1/\beta$, $\mu_y = -0.125$, and $\sigma_y = 0.5$ and let the horizon be $T = 75$. Using these parameter values, solve the consumer's dynamic programming problem by backwards induction. Plot the consumer's value functions $v_t(x)$ and consumption policy functions $c_t(x)$.

(d) How would your answers to (c) change if instead $\alpha = 0.5$? or $\alpha = 2$? How would your answers to (c) change if instead $T = 50$? or $T = 100$? Explain.