Macroeconomics
Chris Edmond

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Problem Set #2: Solutions

1. **Markov chains.** Consider a 2-state Markov chain on x_i, $i = 1, 2$ with transition probabilities $p_{ij} = \text{Prob}[x_{t+1} = x_j \,|\, x_t = x_i]$ for $i, j = 1, 2$ given by the matrix

$$\boldsymbol{P} = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} = \begin{pmatrix} p & 1-p \\ 1-q & q \end{pmatrix}$$

with parameters $p \in [0, 1]$ and $q \in [0, 1]$.

(a) Let λ_i for $i = 1, 2$ denote the eigenvalues of this transition matrix. Solve for λ_i in terms of the parameters p, q. Show that at least one $\lambda_i = 1$. Show that $\max_i |\lambda_i| = 1$. Can there be a zero eigenvalue? Can there be a negative eigenvalue? Explain.

(b) Let $\boldsymbol{\psi}^*$ denote a stationary distribution of the Markov chain. Solve for $\boldsymbol{\psi}^*$ in terms of the parameters p, q. Can there be more than one stationary distribution? Does the sequence of distributions $\boldsymbol{\psi}_{t+1} = \boldsymbol{P}^\top \boldsymbol{\psi}_t$ always converge to such a stationary distribution? Explain.

SOLUTIONS:

(a) The eigenvalues λ_i are given by the roots of the characteristic polynomial

$$f(\lambda) = \lambda^2 - \text{trace}(\boldsymbol{P})\lambda + \det(\boldsymbol{P})$$

where

$$\text{trace}(\boldsymbol{P}) = p_{11} + p_{22} = p + q$$

and

$$\det(\boldsymbol{P}) = p_{11}p_{22} - p_{12}p_{21} = pq - (1-p)(1-q) = p + q - 1$$

That is, the eigenvalues λ_i solve

$$f(\lambda) = \lambda^2 - (p+q)\lambda + (p+q-1) = 0$$

From the quadratic formula, the roots are

$$\begin{aligned}
\{\lambda_1\,,\,\lambda_2\} &= \frac{(p+q) \pm \sqrt{(p+q)^2 - 4(p+q-1)}}{2} \\
&= \frac{(p+q) \pm \sqrt{(p+q)^2 - 4(p+q) + 2^2)}}{2} \\
&= \frac{(p+q) \pm (p+q-2)}{2} \\
&= \{p+q-1\,,\,1\}
\end{aligned}$$

Hence indeed one eigenvalue is $\lambda = 1$ and the other is $\lambda = p + q - 1$. Since both p and q are in $[0, 1]$ the sum $p + q - 1$ is at least -1 and at most $+1$ so indeed $\max_i |\lambda_i| = 1$. There is a zero eigenvalue whenever $p + q - 1 = 0$, i.e., whenever $q = 1 - p$. For example if $p = 1$ and $q = 0$ or the reverse with $p = 0$ and $q = 1$ or if $p = q = 1/2$. Likewise there is a negative eigenvalue whenever $p + q - 1 < 0$, i.e., whenever $q < 1 - p$. In words, this corresponds to a situation where the probability of staying in state 2 is less than the probability of switching from state 1 to state 2.

(b) A stationary distribution $\boldsymbol{\psi}^*$ of the Markov chain satisfies

$$\boldsymbol{\psi}^* = \boldsymbol{P}^\top \boldsymbol{\psi}^*$$

or

$$(\boldsymbol{I} - \boldsymbol{P}^\top)\boldsymbol{\psi}^* = \mathbf{0}$$

For this two state example we have

$$\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} p & 1-q \\ 1-p & q \end{pmatrix}\right]\begin{pmatrix} \psi_1^* \\ \psi_2^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Equivalently

$$(1-p)\psi_1^* + (q-1)\psi_2^* = 0$$

$$(p-1)\psi_1^* + (1-q)\psi_2^* = 0$$

There is only one linearly independent equation here. To pin down the stationary distribution we need to combine this with the adding-up condition $\psi_1^* + \psi_2^* = 1$. This gives

$$\psi_1^* = \frac{(1-q)}{(1-q) + (1-p)}$$

$$\psi_2^* = \frac{(1-p)}{(1-q) + (1-p)}$$

In performing these calculations, we implicitly assumed that $(1+q) + (1-p) \neq 0$, i.e., we do not have both $p = 1$ and $q = 1$ so that we do not have *both* eigenvalues $\lambda = 1$. If instead $p = q = 1$, the Markov transition matrix is $\boldsymbol{P} = \boldsymbol{I}$ and the sequence of distributions is given by $\boldsymbol{\psi}_{t+1} = \boldsymbol{P}^\top \boldsymbol{\psi}_t = \boldsymbol{\psi}_t$ i.e., $\boldsymbol{\psi}_{t+1} = \boldsymbol{\psi}_t$. In this case, any $\boldsymbol{\psi}$ (satisfying $\psi_i \in [0, 1]$ with $\sum_i \psi_i = 1$) is a stationary distribution. By analogy to a scalar linear difference equation $x_{t+1} = ax_t$, this corresponds to $a = 1$.

Even if there is a unique stationary distribution, i.e., if at least one of p or q is not 1, then iterating on $\boldsymbol{\psi}_{t+1} = \boldsymbol{P}^\top \boldsymbol{\psi}_t$ need not converge to that stationary distribution. For example, if $p = q = 0$ such that the other eigenvalue is $p + q - 1 = -1$ then there is a unique stationary distribution $\psi_1^* = \psi_2^* = 1/2$ but the sequence $\boldsymbol{\psi}_{t+1} = \boldsymbol{P}^\top \boldsymbol{\psi}_t$ follows a 2-cycle and never converges to that distribution (unless by chance it starts there). By analogy to a scalar difference equation $x_{t+1} = ax_t$, this corresponds to $a = -1$.

To summarize, to guarantee the existence of a unique stable stationary distribution we need both p, q in $(0, 1)$. This is true more generally. To guarantee the existence of a unique stable stationary distribution for an n-state Markov chain with elements p_{ij} we need all p_{ij} in $(0, 1)$ so that the probability mass is never completely trapped in one state.

2. **Stochastic growth with elastic labor supply.** Suppose the planner seeks to maximize

$$\mathbb{E}\left\{\sum_{t=0}^{\infty} \beta^t u(c_t, l_t)\right\}, \qquad 0 < \beta < 1$$

subject to the resource constraint

$$c_t + k_{t+1} = z_t f(k_t, l_t) + (1 - \delta)k_t, \qquad 0 < \delta < 1$$

with initial conditions $k_0 > 0$ and $z_0 > 0$. Productivity z_t evolves according to a Markov process with transition probabilities $F(z' \,|\, z) = \text{Prob}[z_{t+1} \leq z' \,|\, z_t = z]$ and unconditional mean $\bar{z} > 0$.

In this problem the planner chooses how much labor l_t to supply. Assume that $u(c_t, l_t)$ is strictly increasing, strictly concave in c_t and strictly decreasing, strictly convex in l_t. The production function $f(k_t, l_t)$ is strictly increasing and strictly concave in both arguments and satisfies constant returns to scale.

(a) Let $v(k, z)$ denote the planner's value function. Setup and explain the Bellman equation that determines $v(k, z)$.

(b) Derive the planner's optimality conditions for consumption, capital and labor.

Now suppose that the utility function has the form

$$u(c, l) = \log c - \frac{l^{1+\varphi}}{1+\varphi}, \qquad \varphi > 0$$

and the production function has the form

$$f(k, l) = k^{\alpha} l^{1-\alpha}, \qquad 0 < \alpha < 1$$

(c) Solve for the non-stochastic steady state values of consumption, capital, and labor in terms of model parameters. Suppose there is a permanent increase in the level of productivity $\bar{z}$. Explain how this changes the steady state values of consumption, capital, and labor. Give economic intuition for your answers.

Now suppose that productivity is given by a stationary AR(1) process in logs

$$\log z_{t+1} = (1 - \phi) \log \bar{z} + \phi \log z_t + \varepsilon_{t+1}, \qquad 0 < \phi < 1$$

where the innovations ε_t are IID $N(0, \sigma_\varepsilon^2)$.

(d) Let $\alpha = 0.3$, $\beta = 1/1.05$, $\delta = 0.05$, $\phi = 0.97$, $\varphi = 1$, $\bar{z} = 1$ and $\sigma_\varepsilon = 0.025$. Using these parameter values, use collocation methods to solve the model. In particular, use cubic splines with 99 breakpoints and discretize the shock process using 29 points for productivity z_t and 15 points for the innovations ε_t.

(e) Suppose the economy is at its non-stochastic steady state and that at $t = 0$ there is a 1 standard deviation innovation to productivity, i.e., $\varepsilon_0 = \sigma_\varepsilon = 0.025$. Use the functions you computed in part (d) to calculate and plot impulse responses for the log-deviations (from steady state) of consumption, capital, labor and output for $T = 250$ periods after the shock. Explain your findings.

(f) Simulate a sequence of productivity z_t of length $T = 1{,}000$ starting from $z_0 = \bar{z} = 1$. Use this simulated sequence of productivity and the functions you computed in part (c) to generate simulated sequences of the log-deviations (from steady-state) of consumption, capital, labor and output starting from $k_0 = \bar{k}$. Which of these variables move most closely together? Which of these variables is most volatile? Explain.

(g) How would your answers to (e) and (f) change if φ was much lower, say $\varphi = 0.1$? What about $\varphi = 10$? Give economic intuition for your answers.

(h) Suppose that capital was not needed for production, $\alpha \to 0$. Explain how this simplifies the determination of equilibrium consumption and employment. Explain the implications of this for fluctuations in consumption and employment. What does this suggest about the importance of capital in this model?

Solutions:

(a) The Bellman equation for this problem can be written

$$v(k, z) = \max_{c,l,k'} \left[u(c, l) + \beta \int v(k', z')\, dF(z' \,|\, z) \right]$$

subject to

$$c + k' = z f(k, l) + (1 - \delta)k$$

The Bellman equation characterizes the value $v(k, z)$ of being endowed with k units of capital at the beginning of the period, when the current productivity shock is z, and then proceeding optimally.

(b) Using the resource constraint to eliminate c from the objective, the RHS of the Bellman equation is

$$u(\, z f(k, l) + (1 - \delta)k - k'\, , l) + \beta \int v(k', z')\, dF(z' \,|\, z)$$

The planner's first order condition with respect to l is

$$u_c(c, l)(z f_l(k, l)) + u_l(c, l) = 0$$

The planner's first order condition with respect to k' is

$$-u_c(c, l) + \beta \int v_k(k', z')\, dF(z' \,|\, z) = 0$$

where the subscripts indicate partial derivatives and where it is understood that c satisfies the resource constraint. The l condition can be written

$$-\frac{u_l(c, l)}{u_c(c, l)} = z f_l(k, l) \tag{1}$$

which says the planner equates the marginal rate of substitution between labor and consumption to the marginal product of labor (i.e., the marginal rate of transformation between labor and consumption). The envelope condition is

$$v_k(k,c) = u_c(c,l)(zf_k(k,l) + (1-\delta))$$

Combining the envelope condition with the first order condition for k' gives the consumption Euler equation

$$u_c(c,l) = \beta \int u_c(c',l')(z' f_k(k',l) + (1-\delta))\, dF(z' \,|\, z) \tag{2}$$

where again it is understood that c and c' satisfy the respective resource constraints

$$c + k' = zf(k,l) + (1-\delta)k \tag{3}$$

Equations (1), (2) and (3) are the planner's key optimality conditions and pin down the planner's choice of c, l, k' given the current state k, z.

(c) First note that with these functional forms we have

$$u_c(c,l) = \frac{1}{c}, \qquad u_l(c,l) = -l^{\varphi}$$

and

$$f_l(k,l) = (1-\alpha)k^{\alpha}l^{-\alpha} = (1-\alpha)(k/l)^{\alpha}, \qquad f_k(k,l) = \alpha k^{\alpha-1}l^{1-\alpha} = \alpha(k/l)^{\alpha-1}$$

So the static labor condition becomes

$$cl^{\varphi} = z(1-\alpha)(k/l)^{\alpha}$$

and the consumption Euler equation becomes

$$\frac{1}{c} = \beta \int \frac{1}{c'}\left(z'\alpha(k'/l')^{\alpha-1} + (1-\delta)\right) dF(z' \,|\, z)$$

In a non-stochastic steady state we have $z = z' = \bar{z}$ with certainty and we look for $\bar{c}, \bar{l}, \bar{k}$ that satisfy $c = c' = \bar{c}$, $l = l' = \bar{l}$ and $k = k' = \bar{k}$. Our three key conditions become

$$\bar{c}\bar{l}^{\varphi} = \bar{z}(1-\alpha)(\bar{k}/\bar{l})^{\alpha} \tag{1'}$$

$$1 = \beta\left(\bar{z}\alpha(\bar{k}/\bar{l})^{\alpha-1} + (1-\delta)\right) \tag{2'}$$

and

$$\bar{c} + \delta\bar{k} = \bar{z}\bar{k}^{\alpha}\bar{l}^{1-\alpha} = \bar{y} \tag{3'}$$

These can be solved recursively as follows. First, from the steady-state consumption Euler equation we have the capital/labor ratio

$$\frac{\bar{k}}{\bar{l}} = \left(\frac{\alpha\bar{z}}{\rho+\delta}\right)^{\frac{1}{1-\alpha}}, \qquad \rho \equiv \frac{1}{\beta} - 1$$

and hence the steady-state output/labor ratio

$$\frac{\bar{y}}{\bar{l}} = \bar{z}^{\frac{1}{1-\alpha}} \left(\frac{\alpha}{\rho+\delta}\right)^{\frac{\alpha}{1-\alpha}}$$

Hence the steady-state capital/output ratio is

$$\frac{\bar{k}}{\bar{y}} = \frac{\alpha}{\rho + \delta}$$

independent of $\bar{z}$. Then from the resource constraint the steady-state consumption/output ratio is

$$\frac{\bar{c}}{\bar{y}} = 1 - \delta\frac{\bar{k}}{\bar{y}} = \frac{\rho + (1-\alpha)\delta}{\rho + \delta}$$

also independent of $\bar{z}$. Now write the labor market condition as

$$\bar{l}^{\varphi}\bar{c} = \bar{z}(1-\alpha)(\bar{k}/\bar{l})^{\alpha} = (1-\alpha)\frac{\bar{y}}{\bar{l}}$$

so that

$$\bar{l}^{1+\varphi} = \frac{1-\alpha}{\bar{c}/\bar{y}}$$

Hence steady-state labor is

$$\bar{l} = \left(\frac{(1-\alpha)(\rho+\delta)}{\rho + (1-\alpha)\delta}\right)^{\frac{1}{1+\varphi}}$$

again independent of $\bar{z}$. Thus a permanent increase in $\bar{z}$ leaves $\bar{l}$ unchanged. Then since $\bar{y}/\bar{l}$ is increasing in $\bar{z}$, we must have $\bar{y}$ increasing in $\bar{z}$ and since $\bar{c}/\bar{y}$ is independent of $\bar{z}$ it must also be the case that $\bar{c}$ is increasing in $\bar{z}$. Indeed a 1% increase in $\bar{z}$ increases consumption, capital and output by $\frac{1}{1-\alpha} > 1\%$ (there is a direct effect of 1% + the indirect effects of capital deepening, i.e., some of the increase in output is invested in more capital which increases output further).

(d) The attached Matlab code `elastic_labor_collocation.m` solves the model with the given parameters using collocation.

One trick that I used was to eliminate labor l from the RHS of the Bellman equation using the static optimality condition

$$l^{\varphi}c = (1-\alpha)z\left(\frac{k}{l}\right)^{\alpha}$$

to write

$$l(c\,;\,k,z) = \left(\frac{(1-\alpha)zk^{\alpha}}{c}\right)^{\frac{1}{\varphi+\alpha}}$$

so that we can write period utility as

$$u(c\,;\,k,z) = \log c - \frac{l(c\,;\,k,z)^{1+\varphi}}{1+\varphi}$$

and we can write the resource constraint as

$$c + k' = zf(k\,,\,l(c\,;\,k,z)) + (1-\delta)k$$

and the optimization on the RHS can then be taken over c alone.

The value function $v(k,z)$ and the optimal policy functions for capital accumulation $k' = g(k,z)$, consumption $c(k,z)$, and labor $l(k,z)$ are shown in Figure 1 below.

(e) The impulse response functions for consumption c_t, capital k_t, labor l_t and output y_t given a productivity shock of $\varepsilon_0 = 0.025$ are shown in Figure 2 below.

On impact productivity rises by $\varepsilon_0 = 0.025$ and then decays geometrically back to steady state. On impact, labor rises hence output responds by more than 1-for-1 with productivity. Consumption rises by less than 1-for-1 with output with the remainder invested so that physical capital builds up and output returns to steady state more slowly than does productivity.

Crucially, the economy's response to this temporary productivity shock is *not* the same as its response to a permanent productivity shock. The key is that on impact consumption rises *less than 1-for-1* with output so that income effect on labor is not as large as the substitution effect. Of course consumption responds less than 1-for-1 with output precisely because of the accumulation of physical capital that allows the benefits of the temporarily higher productivity to be amplified and smoothed over time.

(f) Simulated time series are shown in Figure 3 below.

For this simulation, the standard deviations of the log of each of the key variables are:

c	k	l	y	z
0.1472	0.1630	0.0136	0.1572	0.1078

Output is more volatile than productivity. Consumption is somewhat smoother than output, labor is much smoother than either.

The correlation matrix for the logs of these key variables is:

	c	k	l	y	z
c	1.0000	0.9792	0.2888	0.9862	0.9685
k	0.9792	1.0000	0.0885	0.9320	0.8977
l	0.2888	0.0885	1.0000	0.4434	0.5181
y	0.9862	0.9320	0.4434	1.0000	0.9964
z	0.9685	0.8977	0.5181	0.9964	1.0000

Notice in particular that consumption, output, and productivity are highly positively correlated, labor is positively correlated but much less so.

(g) With $\varphi = 0.1$, labor supply is highly elastic (the labor supply curve is nearly horizontal) so a given shift in labor demand as triggered by a productivity shock leads to a large employment response. Similarly, with $\varphi = 10$, labor supply is highly inelastic (the labor supply curve is nearly vertical) so a given shift in labor demand as triggered by a productivity shock leads to a very small employment response. For example, with $\varphi = 10$ the standard deviations of the log of each of the key variables are:

c	k	l	y	z
0.1430	0.1568	0.0021	0.1516	0.1078

So indeed in this case employment is almost constant even though productivity is just as volatile as in parts (e) and (f).

(h) With $\alpha \to 0$ capital disappears from the model leaving us with an essentially static model where outcomes are characterized by the labor supply condition

$$l_t^{\varphi} c_t = z_t$$

and the resource constraint

$$c_t = z_t l_t$$

Together these imply $l_t = 1$ independent of z_t and $c_t = z_t$. Thus without capital this model implies constant employment and consumption (and output) that fluctuates *exactly 1-for-1* with productivity. Employment is constant here because with log utility the income and substitution effects of a change in productivity exactly cancel. Thus in this special case, the response of the economy to a temporary productivity shock is essentially the same as the response to a permanent shock. In this sense, capital accumulation in response to a temporary productivity shock is *the* key mechanism of the original model. This capital accumulation occurs because of intertemporal consumption smoothing and implies that consumption responds *less than 1-for-1* with output so that the substitution effect on labor supply dominates the income effect and employment rises. Without this, as seen here with $\alpha \to 0$, employment does not respond to temporary productivity shocks, just as it doesn't respond to permanent productivity shocks. Put differently, it is intertemporal consumption smoothing that makes the response to temporary productivity shocks different from the response to permanent productivity shocks.

Figure 1: Value function and policy functions

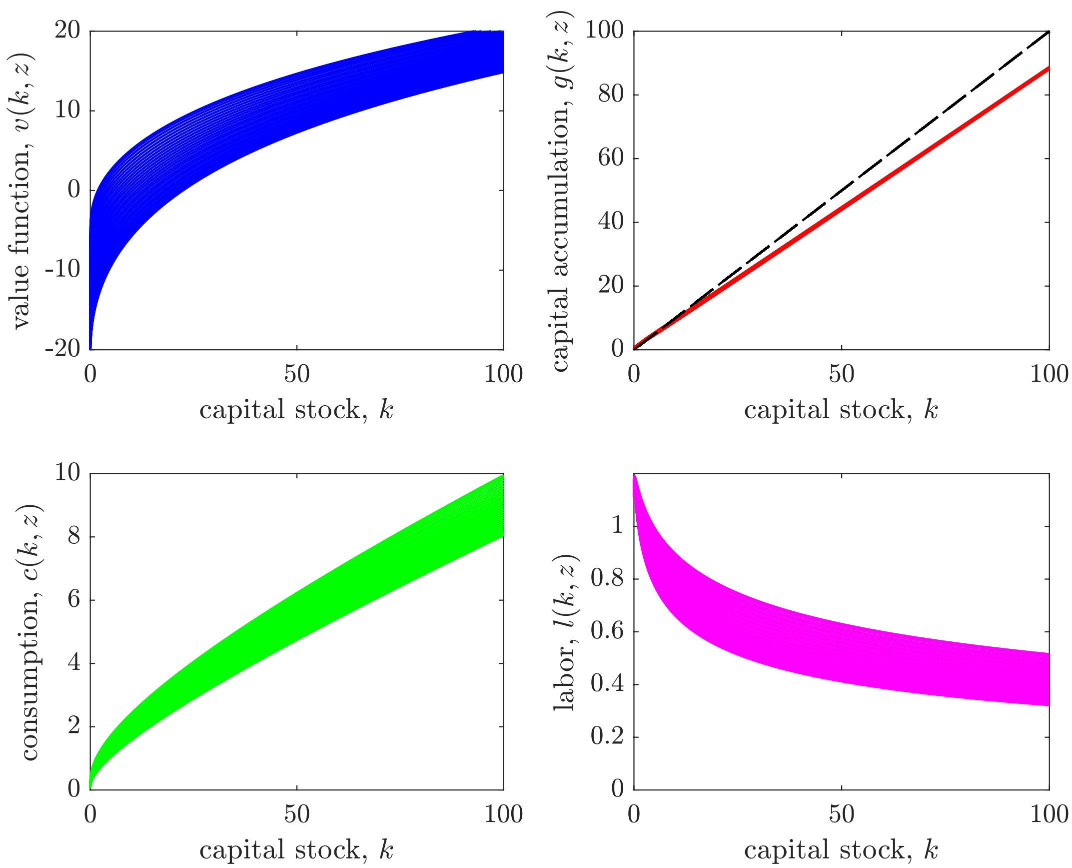

Figure 2: Impulse response functions

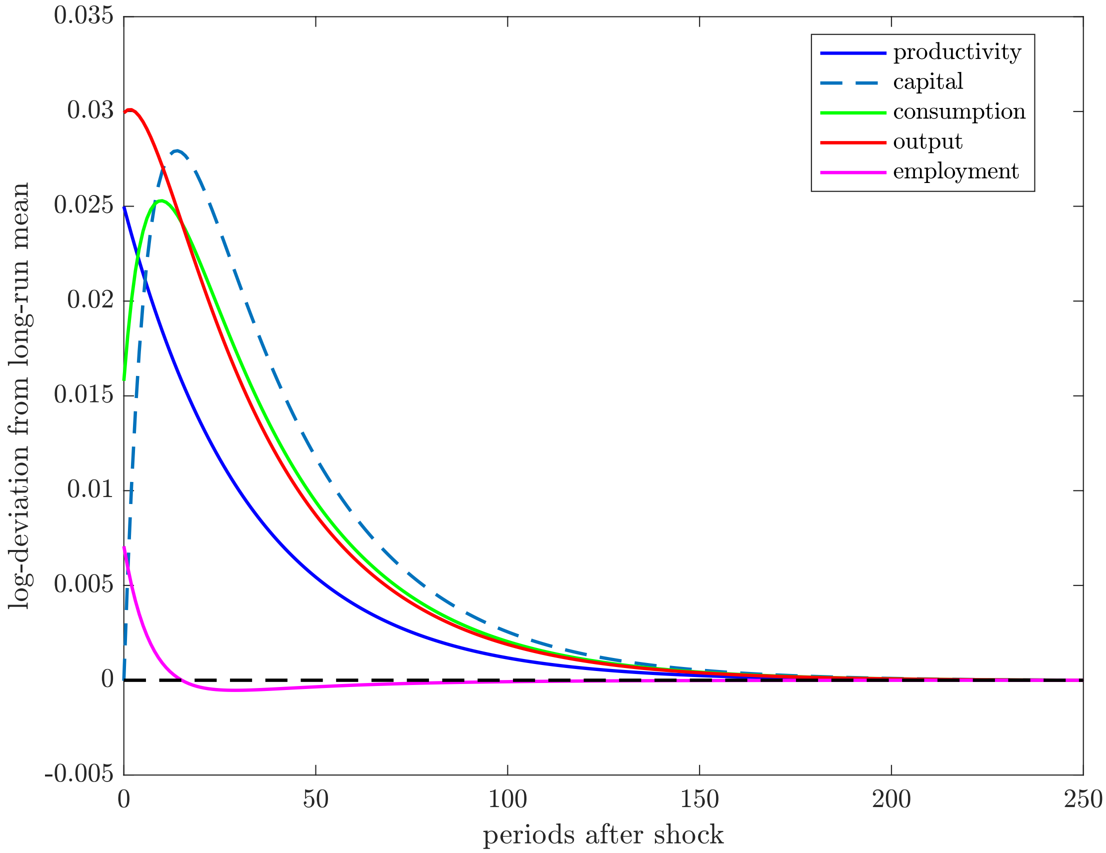

Figure 3: Simulated time series