Faculty of Business and Economics
Department of Economics
THE UNIVERSITY OF MELBOURNE

Macroeconomics
Chris Edmond

Macroeconomics
Problem Set #1: Solutions

1. **Simple difference equations.** Consider the linear difference equation

$$x_{t+1} = \bar{x} + a\,(x_t - \bar{x}), \qquad t = 0, 1, 2, \ldots, \qquad x_0 \in \mathbb{R} \text{ given}$$

   (a) Give a complete account of the possible dynamics of x_t implied by this linear difference equation. Explain how these dynamics depend on the value of the parameter a. Do these dynamics depend on the value of the initial condition x_0? Explain.

   Now consider the nonlinear difference equation

$$x_{t+1} = a\,x_t(1 - x_t), \qquad t = 0, 1, 2, \ldots, \qquad x_0 \in [0, 1] \text{ given}, \qquad a \in (0, 4]$$

   (b) Show that, for this difference equation, x_t lies in $[0, 1]$ for all t.

   (c) How many steady states does this difference equation have? How do these depend on the parameter a?

   (d) Give as complete an account as you can of the possible dynamics of x_t implied by this difference equation. Explain how these dynamics depend on the value of the parameter a. Do these dynamics depend on the value of the initial condition x_0? Explain.

   Hint: Consider the special cases

$$a \in \{\, 0.5\,,\ 1.5\,,\ 2.5\,,\ 3.0\,,\ 3.5\,,\ 4.0 \,\}$$

Solutions:

   (a) Iterating on the difference equation gives

$$\begin{aligned} x_1 - \bar{x} &= a\,(x_0 - \bar{x}) \\ x_2 - \bar{x} &= a\,(x_1 - \bar{x}) = a^2\,(x_0 - \bar{x}) \\ x_3 - \bar{x} &= a\,(x_2 - \bar{x}) = a^3\,(x_0 - \bar{x}) \\ &\ \vdots \end{aligned}$$

   and so for any $t = 0, 1, 2, \ldots$ and a we have

$$x_t = \bar{x} + a^t(x_0 - \bar{x}) \qquad (*)$$

   Now let us consider how this solution depends on the values of a and x_0. First notice that if by coincidence $x_0 = \bar{x}$ then $x_t = \bar{x}$ (regardless of a) for all t. Now suppose $x_0 \neq \bar{x}$. Then if $|a| < 1$ the sequence x_t given by $(*)$ converges to $\bar{x}$ from any initial x_0. That convergence is monotonic if $a \in (0, 1)$ and oscillatory if $a \in (-1, a)$. That convergence takes place in one step if $a = 0$. If $a > 1$ the sequence x_t given by $(*)$ diverges monotonically to $\pm\infty$ (depending on the sign of $x_0 - \bar{x}$). If $a < -1$ the sequence x_t given by $(*)$ diverges in an explosive series of oscillations. Now what about the knife-edge cases $a = 1$ and $a = -1$? If $a = 1$ we simply have

$$x_t = \bar{x} + (1)^t(x_0 - \bar{x}) = x_0$$

That is $x_t = x_0$ for any x_0. Thus if $a = 1$ every x_0 is a steady state. Graphically, the difference equation lies on top of the 45°-line. Finally, if $a = -1$ we have

$$x_t = \bar{x} + (-1)^t(x_0 - \bar{x})$$

and since $(-1)^t = 1$ for even powers $t = 0, 2, 4, 6\dots$ while $(-1)^t = -1$ for odd powers $t = 1, 3, 5, \dots$ we have that $x_t = x_0$ for $t = 0, 2, 4, 6\dots$ while $x_t = x_1 = 2\bar{x} - x_0$ for $t = 1, 3, 5, \dots$. In other words, the sequence x_t is a *2-cycle* of the form $x_0, x_1, x_0, x_1, x_0, x_1, \dots$. Graphically, the difference equation is a straight line with slope -1 and iterating on the difference equation leads to cycles around $\bar{x}$. Indeed the two points in the cycle average out to $\bar{x}$, that is x_0 and x_1 are such that

$$\frac{1}{2}(x_0 + x_1) = \bar{x}$$

Notice again that if by coincidence $x_0 = \bar{x}$ then $x_1 = \bar{x} = x_0$ so again in this degenerate case x_t remains at $\bar{x}$ regardless of a.

(b) Consider the function

$$f(x) = ax(1 - x) = ax - ax^2, \qquad x \in [0, 1], \qquad a > 0$$

Note that $f'(x) = a - 2ax$ and $f''(x) = -2a < 0$ so $f(x)$ is strictly concave. Hence $f(x)$ has a global maximum at the critical point x^* solving $f'(x^*) = 0$ which is given by

$$x^* = \frac{a}{2a} = \frac{1}{2}$$

Hence

$$f(x) \leq \max_x f(x) = f(x^*) = f(1/2) = a(1/2)(1/2) = a/4$$

Then since $a \leq 4$ we have $f(x) \leq 1$. Hence $f(x) = ax(1 - x) \in [0, 1]$ for any $x \in [0, 1]$. Since we are given $x_0 \in [0, 1]$ we have $x_1 = f(x_0) \in [0, 1]$ and $x_2 = f(x_1) \in [0, 1]$ and more generally $x_{t+1} = f(x_t) \in [0, 1]$ for all $t = 0, 1, \dots$. Hence $x_t \in [0, 1]$ for all $t = 0, 1, \dots$.

(c) Consider constant solutions $x_t = x_{t+1} = x^*$ satisfying

$$x^* = ax^*(1 - x^*)$$

Clearly $x^* = 0$ solves this equation hence $x^* = 0$ is a fixed point of the difference equation $x_{t+1} = ax_t(1 - x_t)$ regardless of a. Now consider $x^* \neq 0$. We can then divide both sides by x^* to get

$$1 = a(1 - x^*)$$

or

$$x^* = \frac{a - 1}{a}$$

But now observe that if $a < 1$ then this solution is negative and so, as we know from part (b), cannot be reached by iterating on $x_{t+1} = ax_t(1 - x_t)$ from any $x_0 \in [0, 1]$. Hence we conclude that if $a \leq 1$ then the difference equation has a unique fixed point at $x^* = 0$ but if $a \in (1, 4]$ then the difference equation has *two* fixed points, one at $x^* = 0$ and the other at $x^* = (a - 1)/a$.

(d) (Sketch) Again let $f(x) = ax(1 - x)$ with $f'(x) = a - 2ax$. Note that $f'(0) = a$ and $f'(x^*) = a - 2ax = a - 2a(a - 1)/a = 2 - a$ and note that $|f'(x^*)| = |2 - a| < 1$ for all $a \in (1, 3)$. Hence if $a \in (0, 1)$ then the fixed point $x^* = 0$ is locally stable while if $a \in (1, 3)$ the fixed point $x^* = 0$ is unstable while the fixed point $x^* = (a - 1)/a$ is locally stable. Indeed if $a \in (0, 1)$ we have that $x_{t+1} = ax_t(1 - x_t) \leq x_t$ so that if $a < 1$ the sequence x_t converges to the fixed point $x^* = 0$ for any $x_0 \in [0, 1]$. Similarly, if $a \in (1, 3)$ we have $x_t \to x^* = (a - 1)/a$ for any $x_0 \in [0, 1]$, in oscillations if $a \in (2, 3)$. To summarise, if $a \in (0, 3)$ the sequence x_t is globally convergent to a fixed point. That fixed point is $x^* = 0$ if $a \leq 1$ but $x^* = (a - 1)/a$ if $a \in (1, 3)$. Things are more complex if $a \geq 3$. At $a = 3$ exactly the dynamics *bifurcate*. For $a \in (3, 4)$ neither $x^* = 0$ nor $x^* = (a - 1)/a$ are stable and the iterates follower more complex dynamics. For values of $a \in (3, 1 + \sqrt{6}) = (3, 3.4495...)$ these orbits form stable *limit cycles* given by the roots of $f^2(x) = x$ where $f^2(x)$ denotes the second iterate $f^2(x) \equiv f(f(x))$, i.e., $x_{t+2} = f^2(x_t)$. For higher values $a > 1 + \sqrt{6} \approx 3.4495$ more complex dynamics emerge. Take a look at the Wikipedia entry for the 'logistic map' for some animated examples. The textbook *Dynamics and Bifurcations* by Hale and Koçak gives a more formal treatment if you're interested. The key point here is that even apparently simple difference equations can give rise to quite complex dynamics.

The attached Matlab file `ps1_question1.m` performs these iterations given values for a and x_0. Figures 1 and 2 below illustrate the cases $a = 0.9$ and $a = 1.5$ for which the dynamics converge to $x^* = 0$ from any $x_0 \in [0, 1]$. Figure 3 illustrates the case $a = 2.5$ for which the dynamics converge to $x^* = (a - 1)/a = 0.6$ for any $x_0 \in [0, 1]$. Figure 4 illustrates the case $a = 3$ for which a stable limit cycle emerges. Figures 5 and 6 illustrate $a = 3.5$ and $a = 3.9$ for which more complex dynamics emerge. Figure 7 shows these cases starting from various initial conditions x_0.

2. **Numerical dynamic programming by value function iteration.** Consider the infinite-horizon growth model. The planner chooses capital stocks k_{t+1} for $t = 0, 1, \ldots$ to maximize

$$\sum_{t=0}^{\infty} \beta^t u(c_t), \qquad 0 < \beta < 1$$

subject to the sequence of resource constraints

$$c_t + k_{t+1} \leq f(k_t) + (1 - \delta)k_t, \qquad 0 < \delta < 1$$

with given initial condition

$$k_0 > 0$$

(a) Let $v(k)$ denote the value function for this problem. Setup and explain the Bellman equation that determines $v(k)$.

Now suppose that the period utility function has the isoelastic form

$$u(c) = \frac{c^{1-\sigma} - 1}{1 - \sigma}, \qquad \sigma > 0$$

and that the production function is

$$f(k) = zk^{\alpha}, \qquad 0 < \alpha < 1$$

(b) Solve for the steady state values c^* and k^*. What is the steady state capital/output ratio? What is the steady state consumption/output ratio? What is the steady state savings rate? How does this compare to the 'golden rule' savings rate for this economy? Explain. How if at all do your answers depend on the value of σ? Explain.

(c) Now let $z = 1$, $\alpha = 0.3$, $\beta = 1/1.05$, $\delta = 0.05$ and $\sigma = 1$. Using these parameter values, discretize the state space on a grid of $n = 1001$ points calculate and plot the value function $v(k)$ on this grid of points. Let $c(k)$ be the associated policy function for consumption. Calculate and plot $c(k)$ for these parameter values. How does the savings behavior implied by this policy function compare to the steady-state savings rate from part (b)? Explain.

(d) Now suppose the economy is at steady state then suddenly at $t = 0$ the productivity level z permanently increases from $z = 1$ to $z' = 1.05$. Calculate and plot the new value function and consumption policy function associated with z'. Explain how these compare to the ones you found in part (c). Calculate and plot the transitional dynamics of the economy as it adjusts to its new long-run values. In particular, calculate and plot the time-paths of capital and consumption until they have converged to their new steady state levels. Use a phase diagram to explain these transitional dynamics.

(e) How if at all would your answers to parts (b) through (d) change if σ was lower, say $\sigma = 0.5$? Or higher, say $\sigma = 2$? Give intuition for your answers.

SOLUTIONS:

(a) The Bellman equation for this problem can be written

$$v(k) = \max_{k'} \left[u(f(k) + (1 - \delta)k - k') + \beta v(k') \right]$$

As usual, the Bellman equation characterizes the value $v(k)$ of being endowed with k units of capital at the beginning of the period and then proceeding optimally. Supposing we have found a $v(k)$ that solves the Bellman equation, the consumption policy function can be recovered as

$$c(k) = \operatorname*{argmax}_{c} \left[u(c) + \beta v(f(k) + (1 - \delta)k - c) \right]$$

(b) The first order condition for this problem can be written

$$u_1(c) = \beta v_1(k')$$

where the subscript 1 indicates the first derivative and where it is understood that $c = f(k) + (1 - \delta)k - k'$. The envelope condition is

$$v_1(k) = u_1(c)(f_1(k) + 1 - \delta)$$

Evaluating the latter at k' gives

$$v_1(k') = u_1(c')(f_1(k') + 1 - \delta)$$

where it is understood that $c' = f(k') + (1-\delta)k' - k''$. Hence we have the consumption Euler equation

$$u_1(c) = \beta u_1(c')(f_1(k') + 1 - \delta)$$

In steady state with $k = k' = k^*$ etc and $c = c' = c^*$ etc this simplifies to

$$1 = \beta(f_1(k^*) + 1 - \delta)$$

which can be solved for k^*. From the resource constraint we then have

$$c^* = f(k^*) - \delta k^*$$

With the given functional forms

$$k^* = \left(\frac{\alpha}{\rho+\delta}\right)^{\frac{1}{1-\alpha}} z^{\frac{1}{1-\alpha}}, \qquad \rho \equiv \frac{1}{\beta} - 1$$

Steady state output is then

$$y^* = \left(\frac{\alpha}{\rho+\delta}\right)^{\frac{\alpha}{1-\alpha}} z^{\frac{1}{1-\alpha}}$$

So the steady state capital/output ratio is

$$\frac{k^*}{y^*} = \frac{\alpha}{\rho+\delta}$$

And the steady state consumption/output ratio is

$$\frac{c^*}{y^*} = 1 - \delta\frac{k^*}{y^*} = \frac{\rho + (1-\alpha)\delta}{\rho+\delta}$$

With steady state saving

$$s^* \equiv 1 - \frac{c^*}{y^*} = \delta\frac{k^*}{y^*} = \frac{\delta\alpha}{\rho+\delta}$$

With the production function $y = zk^\alpha$ the 'golden rule' saving rate that maximizes steady state consumption is given by

$$s_{\text{GR}} = \alpha$$

Which is clearly greater than the steady state savings rate

$$s^* = \alpha\frac{\delta}{\rho+\delta} < \alpha = s_{\text{GR}}$$

Notice that with higher patience, as $\rho \to 0$, the savings rate $s^* \to s_{\text{GR}}$.
Finally notice that because of the additively separable intertemporal utility function $\sum_{t=0}^{\infty}\beta^t u(c_t)$, the period utility function $u(c)$ plays no role in determining any of the steady state values.

(c) With the given parameter values, $k^* = 4.8040$, $y^* = 1.6013$ and $c^* = 1.3611$ so that $c^*/y^* = 0.85$ and $\delta k^*/y^* = 0.15$ (85% of output is consumed, 15% is saved/invested). The capital/output ratio is $k^*/y^* = 3$. The attached Matlab file `ps1_question2.m` performs the value function iteration with the given parameters. The results for the value function $v(k)$ and the consumption policy function $c(k)$ are shown in Figure 8. The consumption policy function $c(k)$ corresponds to the stable arm of the saddle path in the usual phase diagram and goes through the steady state (k^*, c^*) where $k^* = 4.8040$ and $c^* = c(k^*) = 1.3611$. These steady state values are indicated by the dashed lines in the right panel. When the capital stock is low, $k < k^*$, consumption is low relative to steady state $c < c^*$ and hence the savings rate is relatively high (higher than $\delta k^*/y^* = 0.15$). When the capital stock is high, $k > k^*$, consumption is high relative to steady state $c > c^*$ and hence the savings rate is relatively low (lower than $\delta k^*/y^* = 0.15$).

(d) When productivity z increases from $z = 1$ to $z' = 1.05$ the value function $v(k)$ and the consumption policy function $c(k)$ both shift up, as shown in Figure 9. The economy is more productive which leads to higher capital, output, and output. Steady state capital increases to $k^{*\prime} = 5.1508$, steady state output increases to $y^{*\prime} = 1.7169$ and steady state consumption increases to $c^{*\prime} = 1.4594$. To see this in a phase diagram, first note that an increase in z shifts the $\Delta c = 0$ locus to the right and shifts up the $\Delta k = 0$ locus (i.e., the curve $zk^\alpha - \delta k$ shifts up). Thus in the long run consumption, output and capital per worker all increase. On 'impact' the level of consumption immediately jumps *up* to $c(0) > c^*$ on the new stable arm going through the new steady state. The level of output also jumps up on impact because of the change in productivity. Capital does not jump on impact because it is predetermined. On impact, consumption jumps by less than the jump in output with the difference being saved. This increase in savings/investment is what allows the economy to build up a new higher level of capital in the long run. As the economy transitions to its new long run, consumption and output continue to rise with the new higher levels of capital. The transitional dynamics are illustrated roughly in Figure 10. Note the jagged path of c_t — an artifact of the relatively coarse grid in the vicinity of these steady states.

(e) The different values of σ do not affect the steady state (long run) values c^*, k^* but do affect the transitional dynamics around the steady state. Intuitively, if $\sigma = 0.5$, consumption is *highly substitutable* over time — i.e., the intertemporal elasticity of substitution is relatively high, $1/\sigma = 2$. In this case, the consumption smoothing motive is *weak* and the planner instead transitions the economy to its new steady state more quickly than in the benchmark with $\sigma = 1$. Alternatively, if $\sigma = 2$, consumption is *highly complementary* over time — i.e., the intertemporal elasticity of substitution is relatively low, $1/2 = 0.5$. In this case, the consumption smoothing motive is *strong* and the planner smooths consumption over a longer period and the convergence to the new steady state is slower than the benchmark with $\sigma = 1$.

Figure 1: Logistic map: $a = 0.9$

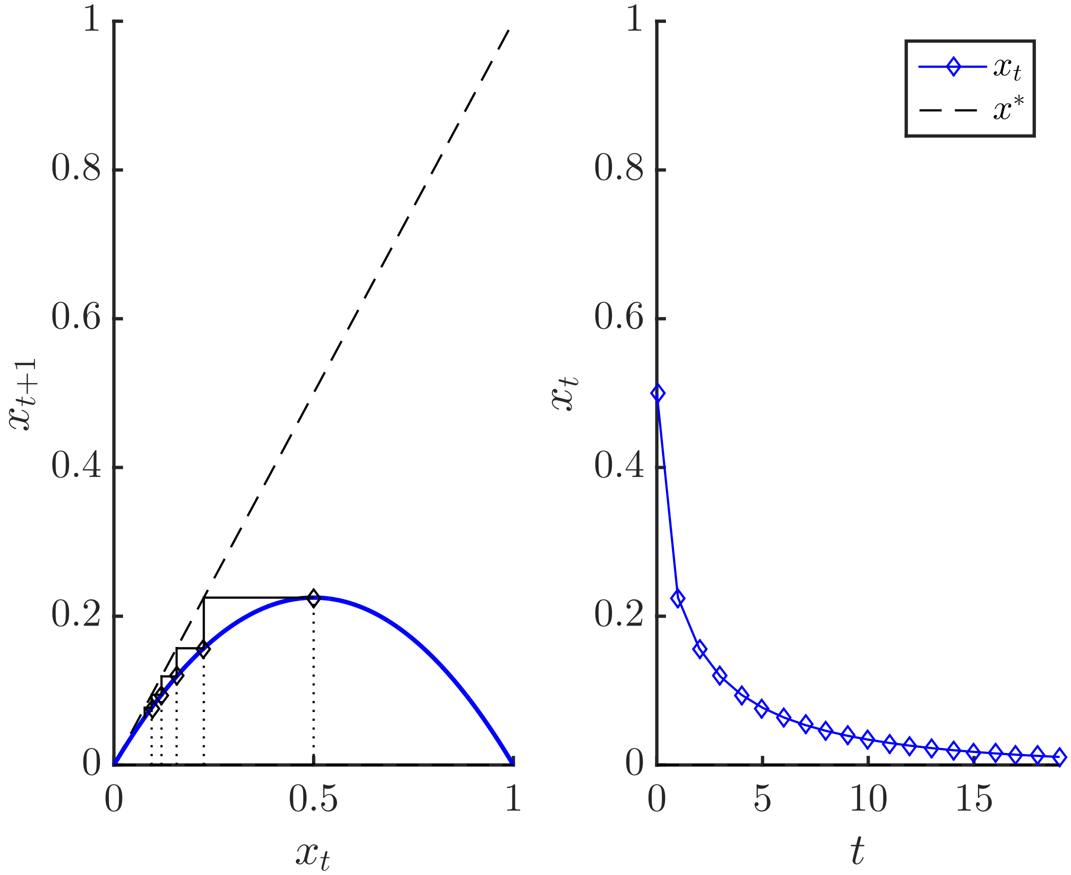

Figure 2: Logistic map: $a = 1.5$

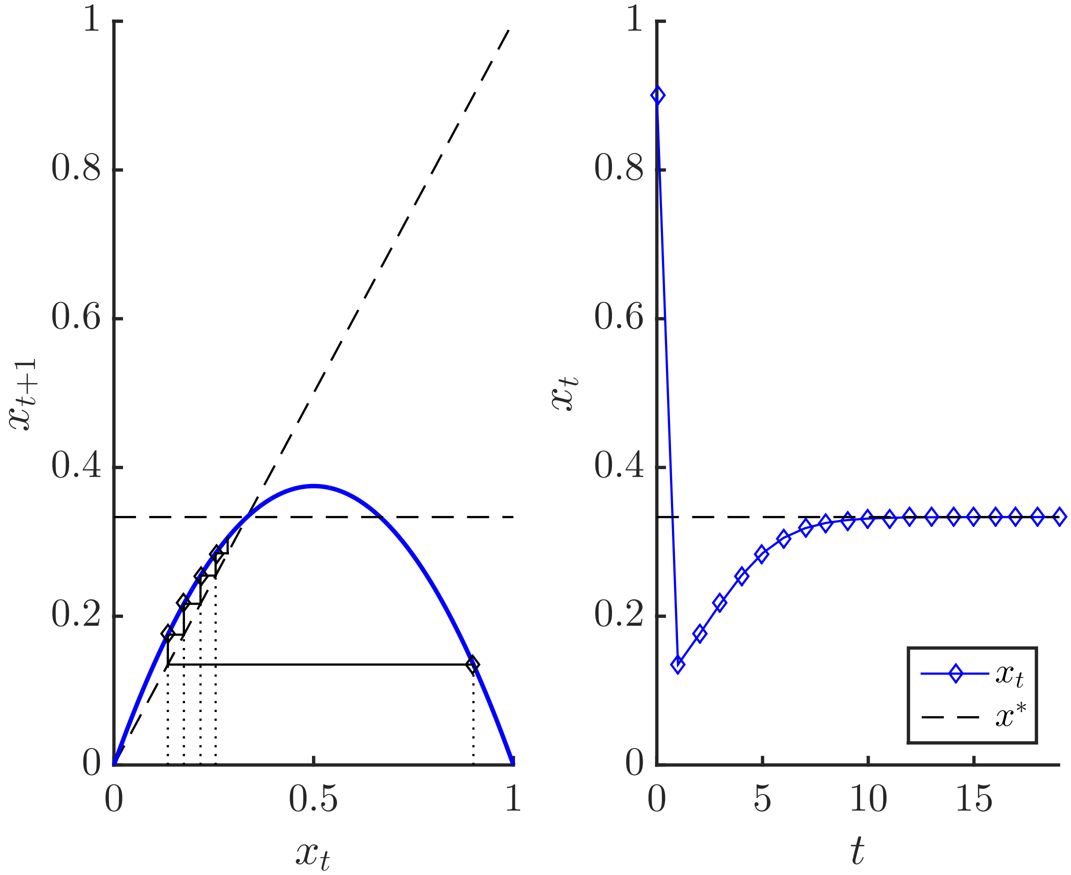

Figure 3: Logistic map: $a = 2.5$

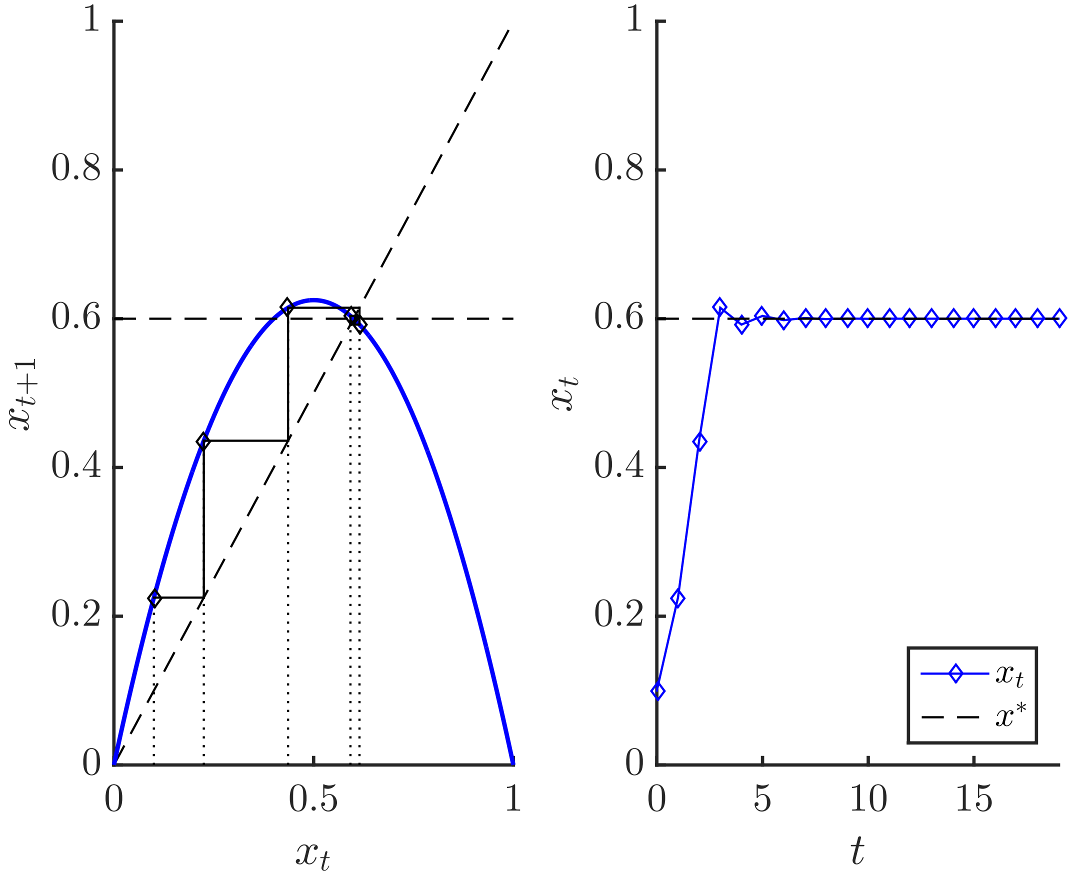

Figure 4: Logistic map: $a = 3.0$

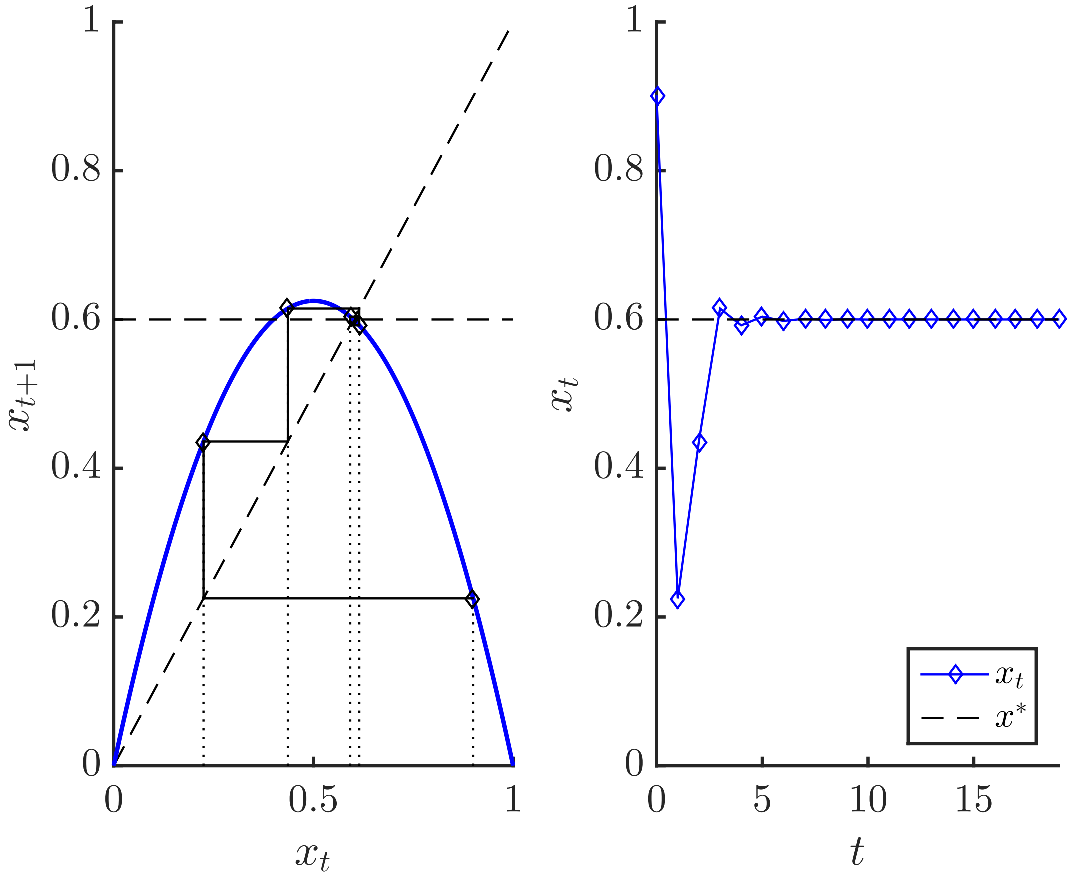

Figure 5: Logistic map: $a = 3.5$

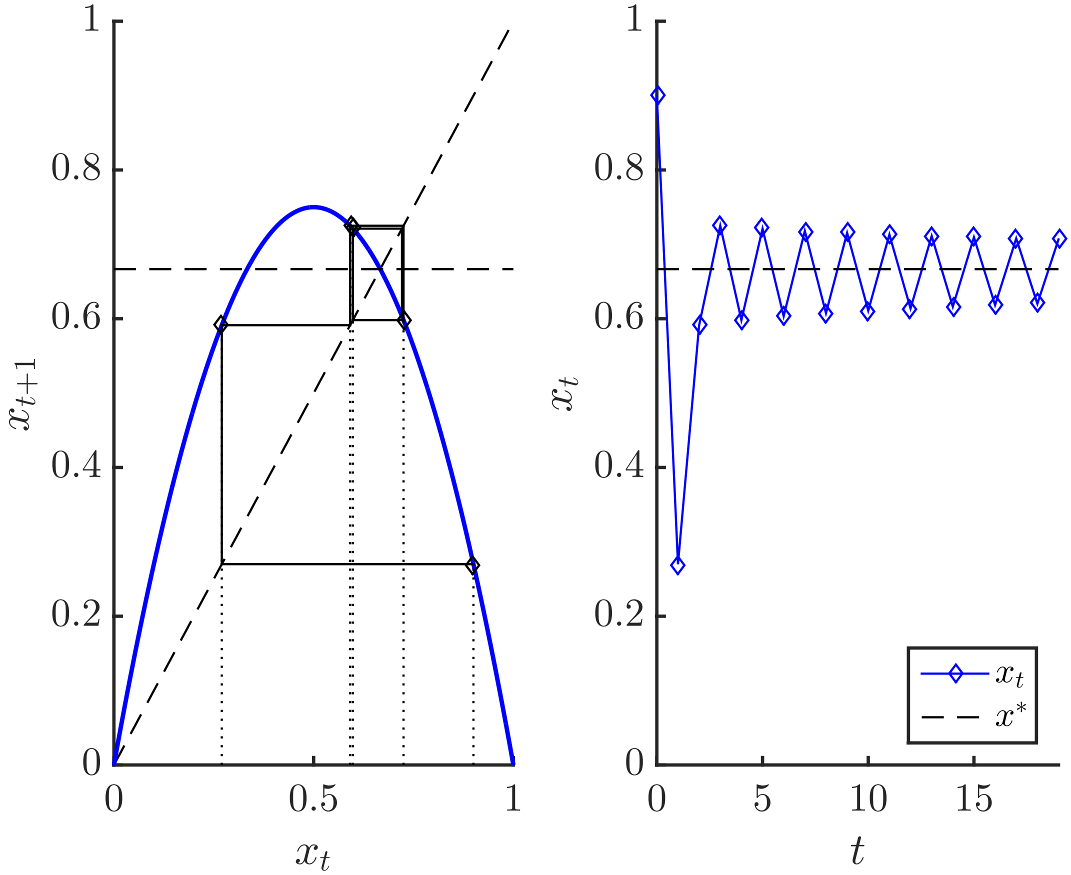

Figure 6: Logistic map: $a = 3.9$

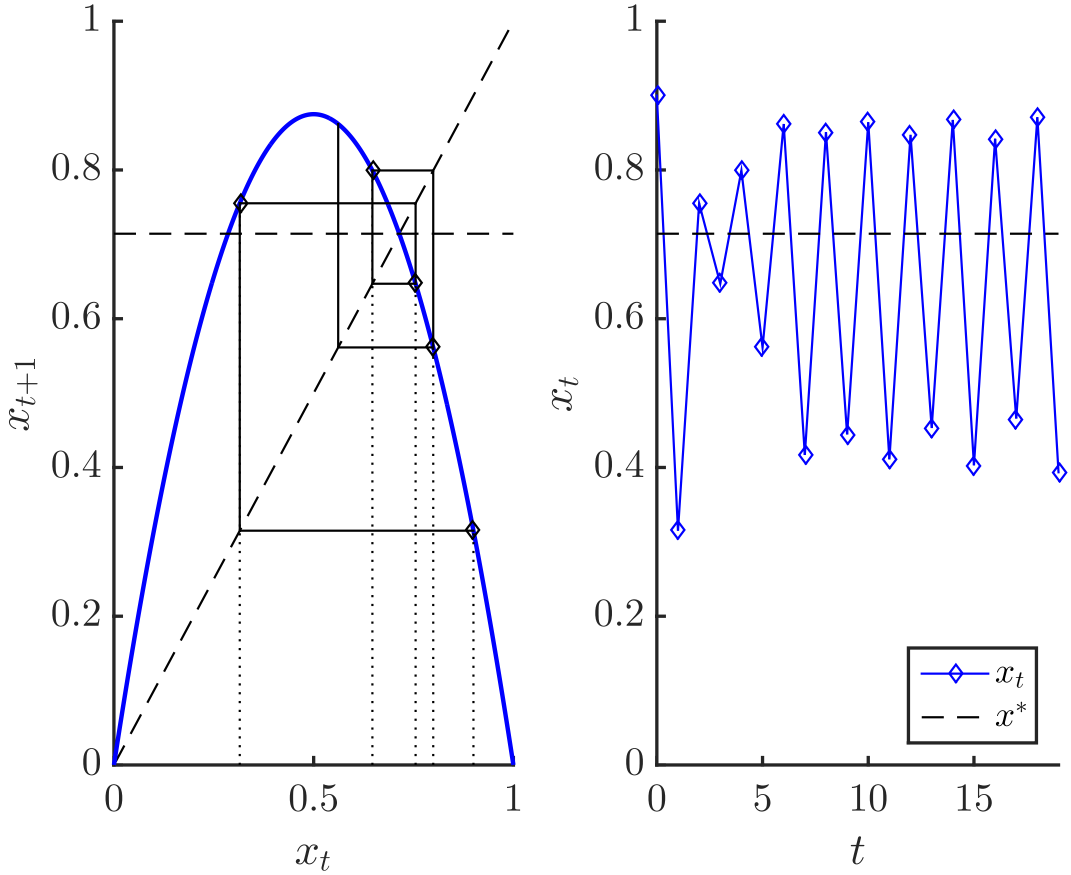

Figure 7: Logistic map: sensitivity to initial conditions x_0

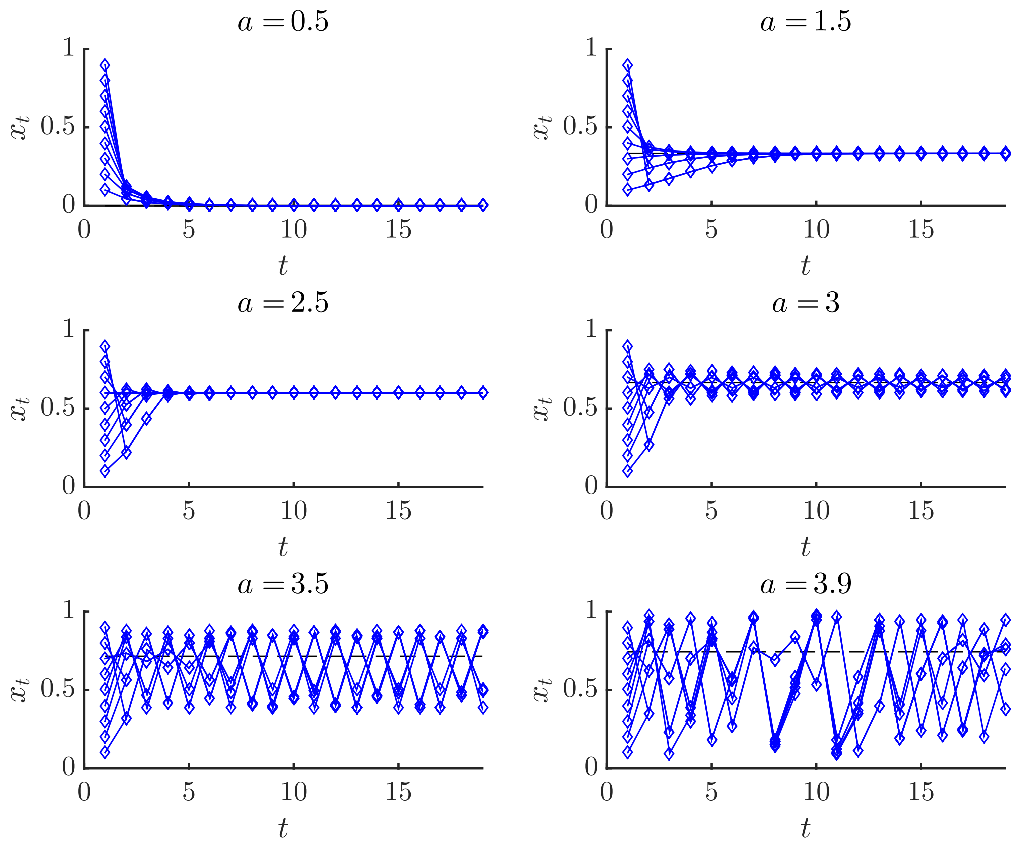

Figure 8: Value function and consumption policy function

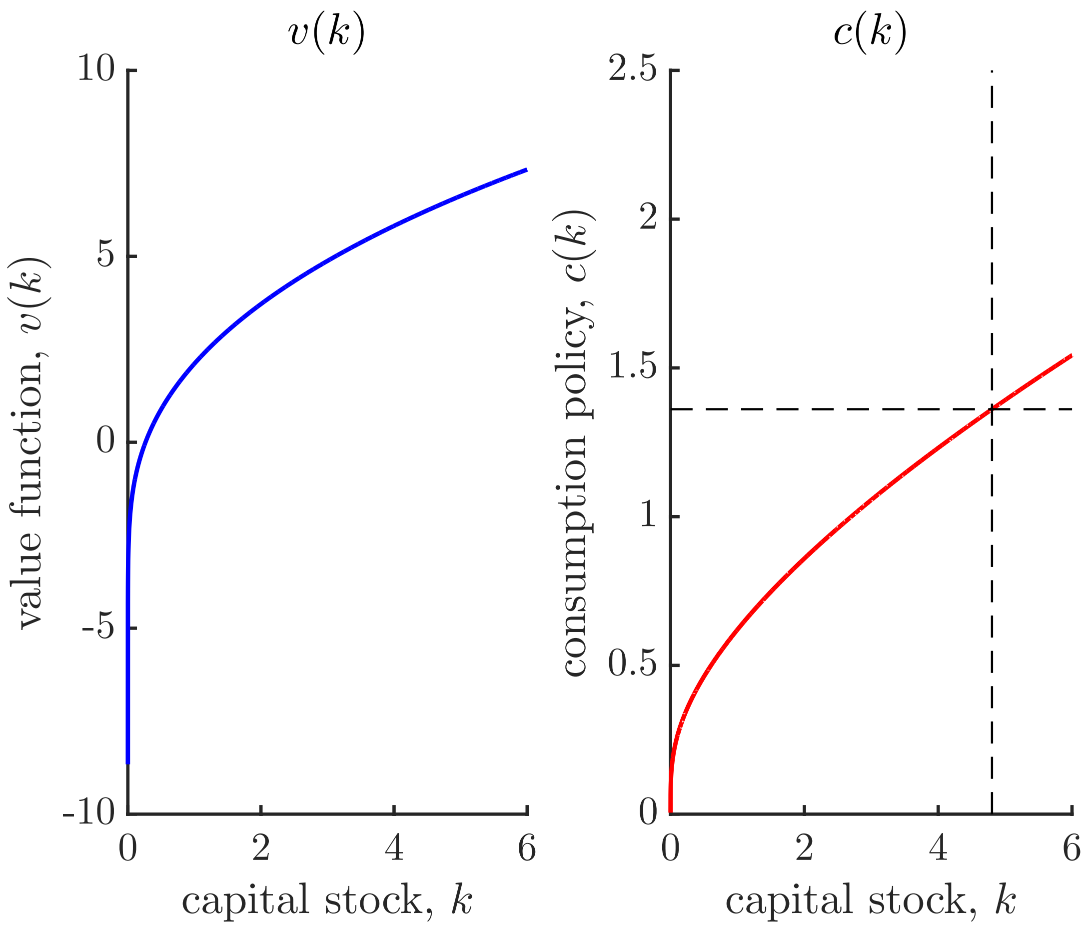

Figure 9: Higher z shifts both $v(k)$ and $c(k)$ up

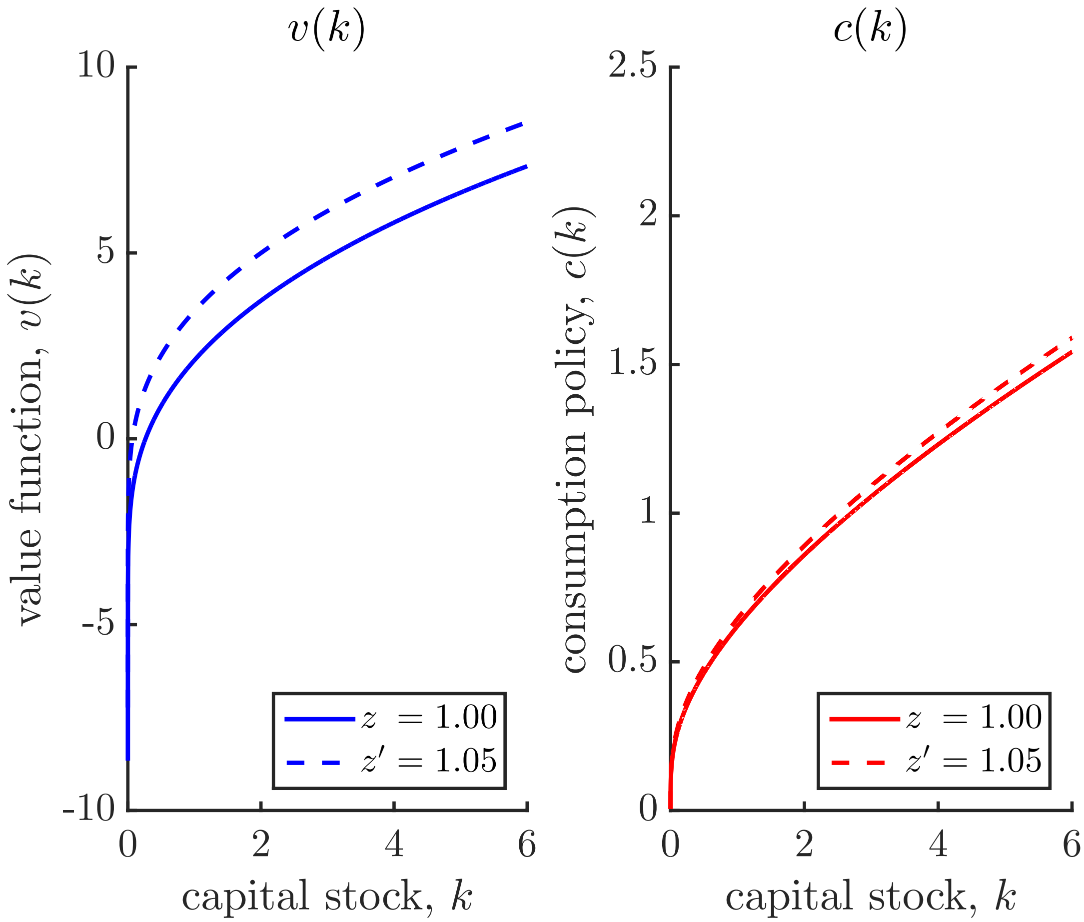

Figure 10: Consumption jumps up on impact