Macroeconomics

Lecture 8: dynamic programming methods, part six

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1st Semester 2019

This class

- Stochastic optimal growth model
  - sequential approach using histories and contingent plans
  - recursive approach using dynamic programming
  - some background on Markov chains

Sequence problem

- Stochastic optimal growth model

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \mathbb{E}\left\{\sum_{t=0}^{\infty} \beta^t u(c_t)\right\}, \qquad 0 < \beta < 1$$

subject to the sequence of constraints, with productivity shock z_t,

$$c_t\,,\ k_{t+1} \geq 0, \qquad \text{and} \qquad c_t + k_{t+1} \leq z_t f(k_t)$$

with the given initial conditions

$$k_0\,,\ z_0 > 0$$

- Problem takes as an input an exogenous *stochastic process* for $\{z_t\}$

- Delivers endogenous stochastic processes $\{c_t\}$ and $\{k_t\}$

Choices

- In the deterministic problem, choose deterministic sequences

- In the stochastic problem, choose stochastic processes that can be interpreted as *contingent plans*

Histories

- Let z^t denote a *history* of realizations of the shock up to and including date t

$$z^t \equiv (z_0\,,\, z_1\,,\, \ldots\,, z_t) = (z^{t-1}\,,\, z_t)$$

- Let $c_t(z^t)$ and $k_{t+1}(z^t)$ denote contingent plans for consumption and capital accumulation conditional on z^t

- History of realizations z^t known at t but unknown as of $t = 0$

- So $c_t(z^t)$ and $k_{t+1}(z^t)$ unknown as of $t = 0$

Expected utility

- Outcomes are ranked according to the expected utility criterion

$$\mathbb{E}\left\{\sum_{t=0}^{\infty} \beta^t\, u(c_t)\right\}$$

- Involves taking expectations with respect to the probability distribution of the random variable $\{c_t(z^t)\}_{t=0}^{\infty}$

- For simplicity, let z_t be a discrete random variable and let $\pi_t(z^t)$ denote the probability of z^t as of date $t = 0$. Then

$$\mathbb{E}\left\{\sum_{t=0}^{\infty} \beta^t\, u(c_t)\right\} = \sum_{t=0}^{\infty} \sum_{z^t} \beta^t\, u(c_t(z^t))\, \pi_t(z^t)$$

Sequence problem

- Stochastic optimal growth model restated

$$\max_{\{c_t(z^t),k_{t+1}(z^t)\}_{t=0}^{\infty}} \quad \sum_{t=0}^{\infty}\sum_{z^t}\beta^t\, u(c_t(z^t))\,\pi_t(z^t)$$

subject to the sequence of resource constraints

$$c_t(z^t)+k_{t+1}(z^t)\leq z_t(z^t)f(k_t(z^{t-1})), \qquad \text{for all } z^t$$

and the non-negativity conditions

$$c_t(z^t)\,,\,k_{t+1}(z^t)\geq 0, \qquad \text{for all } z^t$$

- Takes as given the sequence of probabilities $\pi_t(z^t)$, the initial conditions $k_0, z_0 > 0$ etc

Lagrangian approach

- Lagrangian with stochastic multiplier $\lambda_t(z^t) \geq 0$ for each constraint

$$\mathcal{L} = \sum_{t=0}^{\infty} \sum_{z^t} \beta^t u(c_t(z^t))\pi_t(z^t)$$
$$+ \sum_{t=0}^{\infty} \sum_{z^t} \lambda_t(z^t)\left[z_t(z^t) f(k_t(z^{t-1})) - c_t(z^t) - k_{t+1}(z^t)\right]$$

- First order condition for $c_t(z^t)$ can be written

$$\beta^t u'(c_t(z^t))\pi_t(z^t) = \lambda_t(z^t)$$

- First order condition for $k_{t+1}(z^t)$ can be written

$$\lambda_t(z^t) = \sum_{z' \mid z^t} \lambda_{t+1}(z^t, z')\left[z_{t+1}(z^t, z') f'(k_{t+1}(z^t))\right]$$

where the sum is taken over all states z' that immediately follow z^t

Lagrangian approach

- Eliminating the Lagrange multipliers gives

$$u'(c_t(s^t)) = \beta \sum_{z' \,|\, z^t} u'(c_{t+1}(z^t, z'))\left[z_{t+1}(z^t, z') f'(k_{t+1}(z^t))\right] \frac{\pi_{t+1}(z^t, z')}{\pi_t(z^t)}$$

- To interpret this condition, notice that

$$\frac{\pi_{t+1}(z^t, z')}{\pi_t(z^t)} = \mathrm{Prob}[z' \,|\, z^t]$$

- This is the *conditional probability* of $z_{t+1} = z'$ given the history z^t

- Thus RHS involves a *conditional expectation*

Consumption Euler equation

- In more familiar time-series notation, this is just

$$u'(c_t) = \beta \mathbb{E}_t\{u'(c_{t+1}) z_{t+1} f'(k_{t+1})\}$$

- A stochastic version of the consumption Euler equation

Markov processes

- A (first-order) *Markov process* has the property that, conditional on the current z_t, future realizations are independent of z^{t-1}. In this sense, the current z_t is a *sufficient statistic* for the past

- Markov processes are recursive, and so are a natural setting for dynamic programming approaches

- To begin with, let's consider z_t with discrete support, usually referred to as a *Markov chain*

- As we will see, Markov processes with continuous support have a similar structure

Markov chains

- A finite Markov chain is a triple $(\boldsymbol{z}, \boldsymbol{P}, \boldsymbol{\psi}_0)$ where

  $\boldsymbol{z}$ is an n-vector listing the possible states (outcomes) of the chain

  $\boldsymbol{P}$ is an $n \times n$ probability transition matrix

  $\boldsymbol{\psi}_0$ is an n-vector recording the initial distribution over the states

- Restrictions

$$0 \leq p_{ij} \leq 1, \qquad \text{and} \qquad \sum_{j=1}^{n} p_{ij} = 1 \qquad \text{for all } i = 1, ..., n$$

$$0 \leq \psi_{0,i} \leq 1, \qquad \text{and} \qquad \sum_{i=1}^{n} \psi_{0,i} = 1$$

Interpretation

- Consider stochastic process $\{z_t\}$ induced by a Markov chain
- A realization of z_t takes on the value of one of the states in $\boldsymbol{z}$
- Elements p_{ij} of the transition matrix $\boldsymbol{P}$ have interpretation

$$p_{ij} = \text{Prob}[z_{t+1} = z_j \,|\, z_t = z_i]$$

- Elements $\psi_{0,i}$ of the initial distribution $\boldsymbol{\psi}_0$ have interpretation

$$\psi_{0,i} = \text{Prob}[z_0 = z_i]$$

Transitions

- Let the vector $\boldsymbol{\psi}_t$ be the distribution over $\boldsymbol{z}$ at t, with elements

$$\psi_{t,i} = \text{Prob}[z_t = z_i]$$

- Using the transition probabilities gives

$$\psi_{1,i} = \sum_{j=1}^{n} \text{Prob}[z_1 = z_i \,|\, z_0 = z_j] \, \text{Prob}[z_0 = z_j]$$

$$\vdots$$

$$\psi_{t+1,i} = \sum_{j=1}^{n} \text{Prob}[z_{t+1} = z_i \,|\, z_t = z_j] \, \text{Prob}[z_t = z_j]$$

Transitions

- Collecting these together in matrix notation, we see that

$$\begin{aligned}\boldsymbol{\psi}_1 &= \boldsymbol{P}^\top \boldsymbol{\psi}_0 \\ &\vdots \\ \boldsymbol{\psi}_{t+1} &= \boldsymbol{P}^\top \boldsymbol{\psi}_t, \qquad t = 0, 1, \ldots\end{aligned}$$

where $\boldsymbol{P}^\top$ denotes the transpose of $\boldsymbol{P}$

- Evolves according to a deterministic difference equation

- Iterating forward from date $t = 0$ we have

$$\boldsymbol{\psi}_t = (\boldsymbol{P}^\top)^t \boldsymbol{\psi}_0$$

Stationary distributions

- Stationary distribution $\boldsymbol{\psi}^*$ of Markov chain satisfies

$$\boldsymbol{\psi}^* = \boldsymbol{P}^\top \boldsymbol{\psi}^*$$

(i.e., a fixed point of the difference equation $\boldsymbol{\psi}_{t+1} = \boldsymbol{P}^\top \boldsymbol{\psi}_t$)

- Writing this as

$$(\boldsymbol{I} - \boldsymbol{P}^\top)\boldsymbol{\psi}^* = \mathbf{0}$$

we see $\boldsymbol{\psi}^*$ is an *eigenvector* of $\boldsymbol{P}^\top$ associated with a *unit-eigenvalue*

- Requirement that $\sum_i \psi_i^* = 1$ is a normalization of the eigenvector

Uniqueness and stability (sketch)

- Generally $\boldsymbol{P}^\top$ has n eigenvalues
- Since $\boldsymbol{P}$ is a transition matrix, $\boldsymbol{P}^\top$ has *at least one* unit-eigenvalue
- But may have *multiple* unit-eigenvalues, hence multiple stationary distributions
- Moreover even if there is a unique stationary distribution, iterates $\boldsymbol{\psi}_{t+1} = \boldsymbol{P}^\top \boldsymbol{\psi}_t$ may not converge to it
- A *sufficient condition* for a unique stable stationary distribution is that $0 < p_{ij} < 1$ for all i, j

2×2 example

- Consider two state Markov chain with transition matrix

$$\boldsymbol{P} = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}$$

- Stationary distribution solves (note the transpose)

$$\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1-p & q \\ p & 1-q \end{pmatrix} \right] \begin{pmatrix} \psi_1^* \\ \psi_2^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- Gives

$$\begin{pmatrix} \psi_1^* \\ \psi_2^* \end{pmatrix} = \begin{pmatrix} \dfrac{q}{p+q} \\ \dfrac{p}{p+q} \end{pmatrix}$$

(e.g., $q \to 0$ makes state 2 *absorbing* and state 1 *transient*, etc)

Markov chains with continuous support

- We can also consider Markov chains with *continuous support*

- Suppose z_t has continuous support with density $\psi_t(z)$

- Intuitively

$$\psi_{t+1}(z') = \int p(z' \,|\, z)\psi_t(z)\, dz$$

where $p(z' \,|\, z)$ is density for $z_{t+1} = z'$ conditional on $z_t = z$

- A stationary density $\psi^*(z)$ satisfies the fixed point condition

$$\psi^*(z') = \int p(z' \,|\, z)\psi^*(z)\, dz$$

- There is an analogous theory of uniqueness, stability etc for $\psi^*(z)$

AR(1) example

- Suppose $\{z_t\}$ is a linear Gaussian AR(1) process

$$z_{t+1} = (1-\rho)\mu + \rho z_t + \sigma \varepsilon_{t+1}, \qquad \varepsilon_{t+1} \sim \text{IID } N(0,1)$$

- Then

$$p(z' \mid z) = \frac{1}{\sigma} \phi \left(\frac{z' - (1-\rho)\mu - \rho z}{\sigma} \right)$$

where $\phi(\varepsilon)$ is the PDF of the standard normal distribution

$$\phi(\varepsilon) \equiv \frac{1}{\sqrt{2\pi}} e^{-\varepsilon^2/2}$$

AR(1) example

- If $|\rho| < 1$, then a unique, stable stationary density

  $$\psi^*(z) = \frac{1}{\sigma^*}\phi\left(\frac{z-\mu}{\sigma^*}\right)$$

  where

  $$\sigma^* = \frac{\sigma}{\sqrt{1-\rho^2}}$$

Stochastic dynamic programming

- Suppose z_t is first-order Markov with conditional density

$$\pi(z' \,|\, z)$$

- Bellman equation for this problem

$$v(k,z) = \max_{k'} \left[u(zf(k) - k') + \beta \int v(k',z')\, \pi(z' \,|\, z)\, dz' \right]$$

- First order condition for k'

$$u'(zf(k) - k') = \beta \int v_k(k',z')\, \pi(z' \,|\, z)\, dz'$$

- Envelope condition

$$v_k(k,z) = u'(zf(k) - k')zf'(k)$$

Stochastic dynamic programming

- Eliminating $v_k(k', z')$ using the envelope condition then gives

$$u'(zf(k) - k') = \beta \int u'(z'f(k') - k'')z'f'(k')\, \pi(z' \,|\, z)\, dz'$$

which, in our usual time-series notation, is just

$$u'(c_t) = \beta \mathbb{E}_t\{u'(c_{t+1})z_{t+1}f'(k_{t+1})\}$$

where it is understood that $c_t = z_t f(k_t) - k_{t+1}$ etc.

Stochastic dynamic programming

- Let $k' = g(k, z)$ be the optimal policy that solves this dynamic programming problem

- This is a stochastic difference equation of the form

$$k_{t+1} = g(k_t, z_t)$$

- We cannot expect $\{k_t\}$ to converge to some steady state k^*

- What about the *distribution* of k?

IID example

- Suppose policy function has the multiplicative form $k_{t+1} = z_t g(k_t)$ and that z_t is IID over time with cumulative distribution

$$H(z) \equiv \text{Prob}[\, z_t \leq z \,]$$

- Now consider the cumulative distribution of k at time t

$$\Psi_t(k) \equiv \text{Prob}[\, k_t \leq k \,]$$

- For example, for $t = 1$ we have

$$\begin{aligned} \Psi_1(k) &= \text{Prob}[\, k_1 \leq k \,] \\ &= \text{Prob}[\, z_0 g(k_0) \leq k \,] = \text{Prob}\Big[\, z_0 \leq \frac{k}{g(k_0)} \,\Big] \\ &= H\Big(\frac{k}{g(k_0)}\Big) \end{aligned}$$

IID example

- Let $P(k' \,|\, k)$ denote the conditional distribution

$$P(k' \,|\, k) \equiv \text{Prob}[\, k_{t+1} \leq k' \,|\, k_t = k]$$

- For this IID example, we have

$$P(k' \,|\, k) = H\Big(\frac{k'}{g(k)}\Big)$$

- Then cumulative distribution of k satisfies the law of motion

$$\Psi_{t+1}(k') = \int P(k' \,|\, k)\, d\Psi_t(k)$$

- So that if there is a density representation

$$\psi_{t+1}(k') = \int p(k' \,|\, k)\, \psi_t(k)\, dk$$

IID example

- A stationary density $\psi^*(k)$ is a fixed point of this law of motion

$$\psi^*(k') = \int p(k' \,|\, k)\, \psi^*(k)\, dk$$

- More generally, we would have a *joint distribution* over the state variables (k, z) induced by (i) the policy function $k' = g(k, z)$ and (ii) the exogenous conditional density $\pi(z' \,|\, z)$

- We would then look for a fixed point for that joint distribution

- We'll see lots of examples of this

Next class

- Practical stochastic dynamic programming problems
  - numerical integration to help compute expectations
  - extending our collocation tricks