Macroeconomics

Lecture 5: dynamic programming methods, part three

Chris Edmond

1st Semester 2019

This class

- Principle of optimality
  - is solving the Bellman equation the same as solving the original sequence problem?
- Properties of the value function [skim this, for reference]
  - using the maximum theorem and the contraction mapping theorem to deduce properties of the value function
  - what conditions are needed for value function to be increasing? concave? differentiable?

Sequence vs. recursive formulations

- General form of the *sequence problem* is to find

$$\sup_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \tag{S}$$

subject to a sequence of *constraint sets*

$$x_{t+1} \in \Gamma(x_t), \qquad t = 0, 1, 2, \ldots$$

and given initial condition

$$x_0 \in X$$

- Corresponding to this is the *recursive problem*

$$v(x) = \sup_{y \in \Gamma(x)} \left[F(x, y) + \beta v(y)\right] \qquad \text{for all } x \in X \tag{R}$$

Principle of optimality

- Do the solutions to these problems coincide?
- General idea is that

  (i) solution v to (R) evaluated at $x = x_0$ gives the sup in (S), and

  (ii) a sequence $\{x_{t+1}\}_{t=0}^{\infty}$ attains the sup in (S) if and only if

$$v(x_t) = F(x_t, x_{t+1}) + \beta v(x_{t+1}), \qquad t = 0, 1, 2, \ldots$$

- Bellman called this general idea the *Principle of Optimality*
- What assumptions on the primitives deliver this result?

Four primitives

- State space $X \subseteq \mathbb{R}^n$

- Correspondence (set-valued function) $\Gamma : X \rightrightarrows X$ describing feasible choices as function of current state $x \in X$

  The *graph* of this correspondence is

  $$A \equiv \text{graph}(\Gamma) = \{\, (x, y) \in X \times X \ : \ y \in \Gamma(x) \,\}$$

- Return function $F : A \to \mathbb{R}$ giving flow payoff

- Discount factor $\beta \geq 0$

Feasible plans

- Let $\pi(x_0)$ denote set of *feasible plans* $\{x_{t+1}\}_{t=0}^{\infty}$ starting from x_0

$$\pi(x_0) = \Big\{ \{x_{t+1}\}_{t=0}^{\infty} \quad : \quad x_{t+1} \in \Gamma(x_t), \quad t = 0, 1, 2, \ldots \Big\}$$

- Let $\boldsymbol{x} \in \pi(x_0)$ denote a typical feasible plan

Ensuring sequence problem is well-defined

- ASSUMPTION 1. Constraint set $\Gamma(x)$ nonempty for all $x \in X$

- ASSUMPTION 2. The objective $\lim_{n\to\infty} \sum_{t=0}^{n} \beta^t F(x_t, x_{t+1})$ exists for all $x_0 \in X$ and all $\boldsymbol{x} \in \pi(x_0)$

  REMARK. A sufficient condition for Assumption 2 is that $F(x, y)$ is bounded and $0 < \beta < 1$. But weaker conditions often work too

Ensuring sequence problem is well-defined

- Now define partial sums of the form

$$u_n(\boldsymbol{x}) \equiv \sum_{t=0}^{n} \beta^t F(x_t, x_{t+1}), \qquad \boldsymbol{x} \in \pi(x_0)$$

  By Assumption 2, we can also define the limit

$$u(\boldsymbol{x}) \equiv \lim_{n\to\infty} u_n(\boldsymbol{x})$$

- This allows us to define a *supremum function* v^* by

$$v^*(x_0) \equiv \sup_{\boldsymbol{x}\in\pi(x_0)} u(\boldsymbol{x}) \tag{S}$$

$(S) \Rightarrow (R)$

- Under Assumptions 1 and 2, the v^* that solves (S) also solves (R)
- What about the converse?
- Suppose v solves (R), does v also solve (S)? That is, does $v = v^*$?

Partial converse

- In general no
- But suppose v solves (R) and satisfies boundedness condition

$$\lim_{n\to\infty} \beta^n v(x_n) = 0, \qquad \text{for all } \boldsymbol{x} \in \pi(x_0) \text{ and } x_0 \in X \tag{B}$$

  Then $v = v^*$

- In other words, v^* solves (R) but (R) may have other solutions
- If (R) has other solutions, they violate boundedness condition (B)

Maximum theorem

Let $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$ and let the function $f : X \times Y \to \mathbb{R}$ be continuous and let the correspondence $\Gamma : X \rightrightarrows Y$ be compact-valued and continuous. Then (i) the function

$$h(x) \equiv \max_{y \in \Gamma(x)} f(x, y)$$

is continuous and (ii) the solution correspondence $G : X \rightrightarrows Y$ given by

$$G(x) \equiv \underset{y \in \Gamma(x)}{\operatorname{argmax}} f(x, y) = \{ y \in \Gamma(x) : f(x, y) = h(x) \}$$

is nonempty, compact-valued and upper hemicontinuous (see appendix)

REMARK. If $f(x, y)$ is strictly concave in y and $\Gamma(x)$ is convex, the solution correspondence is single-valued and we write it $g(x)$. Since $g(x)$ is single valued and upper hemicontinuous, it is continuous

Bounded returns

- Let's now focus on the functional equation

$$v(x) = \max_{y \in \Gamma(x)} \big[F(x,y) + \beta v(y)\big], \qquad \text{for all } x \in X$$

  with some additional structure

- Assumption 3. $X \subseteq \mathbb{R}^n$ is convex and the correspondence $\Gamma : X \rightrightarrows X$ is nonempty, compact-valued and continuous

- Assumption 4. $F : A \to \mathbb{R}$ is bounded, continuous and $0 < \beta < 1$

Bounded returns

- Recall $C(X)$ is set of bounded continuous functions $f : X \to \mathbb{R}$

  We can now conclude that the operator

  $$Tf(x) = \max_{y \in \Gamma(x)} \left[F(x,y) + \beta f(y) \right], \qquad \text{for all } x \in X$$

  maps $C(X)$ to $C(X)$. That is, if $f \in C(X)$ then $Tf \in C(X)$ too

- This is because, under Assumptions 3 and 4:

  – the RHS maximizes a continuous function over a compact-valued, continuous correspondence, hence (i) the maximum is attained, and (ii), by the maximum theorem, Tf is continuous

  – both F and f are bounded, so Tf is bounded

Bounded returns

- So T maps $C(X)$ to $C(X)$
- Moreover T satisfies Blackwell's conditions, so T is a contraction
- Since $C(X)$ is a complete metric space and T is a contraction, there is a unique fixed point $v = Tv \in C(X)$ that solves (R)
- Since Assumptions 1, 2 and the boundedness condition (B) are satisfied, this v also solves (S)
- What else can we say about v?

A useful result

- Let (S, d) be a complete metric space and let $T : S \to S$ be a contraction mapping with fixed point $v \in S$. Then

  (i) if S' is a closed subset of S and $T(S') \subseteq S'$ then $v \in S'$, and

  (ii) if $T(S') \subseteq S'' \subseteq S$, then $v \in S''$

Examples

- Let
  - S denote the set of continuous functions
  - S' denote the set of nondecreasing functions, and
  - S'' denote the set of strictly increasing functions
- Then
  - if T maps $f \in S$ to $Tf \in S'$ then $v \in S'$ so v nondecreasing
  - if T maps $f \in S'$ to $Tf \in S''$ then $v \in S''$ so v strictly increasing

General idea

- Suppose f has some property $\mathcal{P}$ and T preserves this property. Then Tf also has $\mathcal{P}$

- Then by induction $T^n f$ also has has $\mathcal{P}$ for any finite $n = 0, 1, \ldots$

- Then if $\mathcal{P}$ is a property that is preserved by uniform convergence, the fixed point $v = Tv$ also has $\mathcal{P}$

Monotonicity

- Suppose we also have the following *monotonicity* assumptions

- Assumption 5. $F(x, y)$ strictly increasing in x for each y

- Assumption 6. $\Gamma(x)$ increasing in that if $x \leq x'$ then $\Gamma(x) \subseteq \Gamma(x')$

Value function v is strictly increasing

- Suppose the primitives X, Γ, F, β satisfy Assumptions 3, 4, 5 and 6 and suppose v is the unique solution to

$$v(x) = \max_{y \in \Gamma(x)} \left[F(x, y) + \beta v(y)\right], \qquad \text{for all } x \in X$$

- Then the value function v is strictly increasing

Proof (sketch)

- Let $f \in C(X)$ and let $x < x'$. Then we have

$$Tf(x) = \max_{y \in \Gamma(x)} \big[F(x, y) + \beta f(y) \big]$$

$$\leq \max_{y \in \Gamma(x')} \big[F(x, y) + \beta f(y) \big]$$

$$< \max_{y \in \Gamma(x')} \big[F(x', y) + \beta f(y) \big] = Tf(x')$$

- Hence T maps bounded continuous functions f into bounded continuous and strictly increasing functions Tf

Proof (sketch)

- Now let $C'(X)$ denote the set of bounded continuous nondecreasing functions on X and let $C''(X) \subset C'(X)$ denote the set of bounded continuous strictly increasing functions

- Then since T maps $f \in C'(X)$ to $Tf \in C''(X)$ the fixed point $v = Tv \in C''(X)$ too. Hence v is strictly increasing

Curvature

- Assumption 7. F is strictly concave, that is, for any $\theta \in (0,1)$

$$F(\,\theta(x,y) + (1-\theta)(x',y')\,) > \theta F(x,y) + (1-\theta)F(x',y')$$

for any (x,y) and (x',y') in the graph of Γ

- Assumption 8. Γ is convex in the sense that for any $\theta \in (0,1)$

$$y \in \Gamma(x) \text{ and } y' \in \Gamma(x')$$

implies

$$\theta y + (1-\theta)y' \in \Gamma(\,\theta x + (1-\theta)x'\,)$$

for any $x, x' \in X$

(since X is convex, equivalent to assuming graph of Γ is convex)

Value function v is strictly concave

- Suppose the primitives X, Γ, F, β satisfy Assumptions 3, 4 and 7, 8 and suppose v is the unique solution to

$$v(x) = \max_{y \in \Gamma(x)} \left[F(x, y) + \beta v(y)\right], \qquad \text{for all } x \in X$$

- Then v is strictly concave

- Moreover there is a continuous single-valued policy function $y = g(x)$ that attains the maximum

Proof (sketch)

- Let $x \neq x'$ and let $x_\theta = \theta x + (1-\theta)x'$ for $\theta \in (0,1)$
- Let $y \in \Gamma(x)$ be such that $Tf(x) = F(x,y) + \beta f(x)$ and likewise Let $y' \in \Gamma(x')$ be such that $Tf(x') = F(x',y') + \beta f(x')$
- Then

$$
\begin{aligned}
Tf(x_\theta) &\geq F(x_\theta, y_\theta) + \beta f(y_\theta) \\
&> \theta\big[F(x,y) + \beta f(y)\big] + (1-\theta)\big[F(x',y') + \beta f(y')\big] \\
&= \theta Tf(x) + (1-\theta)Tf(x')
\end{aligned}
$$

Hence Tf is strictly concave

Proof (sketch)

- Now let $C'(X)$ denote the set of bounded continuous concave functions on X and let $C''(X) \subset C'(X)$ denote the set of bounded continuous strictly concave functions

- Then since T maps $f \in C'(X)$ to $Tf \in C''(X)$ the fixed point $v = Tv \in C''(X)$ too. Hence v is strictly concave

- Then since the RHS of Tv is strictly concave in y and the constraint set $\Gamma(x)$ is convex, by the maximum theorem there is a unique $y = g(x)$ that attains the maximum

Benveniste-Scheinkman theorem

- Let $X \subseteq \mathbb{R}^n$ be convex and let $v : X \to \mathbb{R}$ be continuous and concave. Fix $x_0 \in D \subseteq X$. If there exists a concave differentiable function $w : D \to \mathbb{R}$ such that

$$w(x_0) = v(x_0)$$

and

$$w(x) \leq v(x) \qquad \text{for all } x \in D$$

- Then v is differentiable at x_0 and

$$\frac{\partial v(x_0)}{\partial x_i} = \frac{\partial w(x_0)}{\partial x_i}$$

Proof (sketch)

- A *supergradient* of a concave function v at x_0 is a vector p such that

$$p \cdot (x - x_0) \geq v(x) - v(x_0) \qquad \text{for all } x \in X$$

Generalizes the notion of a gradient to nondifferentiable functions. For continuous functions there is always at least one supergradient. For a differentiable function, the supergradient is unique

- But then

$$p \cdot (x - x_0) \geq v(x) - v(x_0) \geq w(x) - w(x_0) \qquad \text{for all } x \in D$$

Since p is a supergradient of w and w is differentiable, p is unique. Since p is unique, v is differentiable at x_0

Value function v is differentiable

- Assumption 9. F is continuously differentiable on its interior

- Now suppose the primitives X, Γ, F, β satisfy Assumptions 3, 4 and 7, 8, 9 and suppose v is the unique solution to

$$v(x) = \max_{y \in \Gamma(x)} \left[F(x, y) + \beta v(y)\right], \qquad \text{for all } x \in X$$

  and that $y = g(x)$ is the policy function that attains the maximum

- Then v is continuously differentiable on its interior with

$$\frac{\partial v(x)}{\partial x_i} = \left.\frac{\partial F(x, y)}{\partial x_i}\right|_{y=g(x)}$$

  (cf., the envelope condition)

Proof (sketch)

- Fix some x_0 and define

$$w(x) \equiv F(x, g(x_0)) + \beta v(x_0)$$

- Since F is concave and differentiable, so is w. Moreover note that

$$w(x) \leq \max_{y \in \Gamma(x)} \left[F(x,y) + \beta v(y)\right] = v(x)$$

  with equality at $x = x_0$

- Hence Benveniste-Scheinkman conditions satisfied and v is differentiable at x_0

Next class

- Practical dynamic programming
- Solving dynamic programming problems like this on a computer

Appendix: continuity for correspondences

Consider a correspondence $\Gamma : X \rightrightarrows Y$

Because of the ambiguity of defining the inverse of a correspondence, there are two notions of continuity

A correspondence Γ is *lower hemicontinuous* (lhc) at $x \in X$ if (i) $\Gamma(x)$ is nonempty and (ii) for every $y \in \Gamma(x)$ and sequence $x_n \to x$, there exists $N \geq 1$ and sequence $y_n \to y$ such that $y_n \in \Gamma(x_n)$ for all $n \geq N$

Roughly speaking, Γ is lhc if it avoids 'exploding in the limit' at x and is not lhc if it 'jumps up' at x

Appendix: continuity for correspondences

A compact-valued correspondence Γ is *upper hemicontinuous* (uhc) at $x \in X$ if (i) $\Gamma(x)$ is nonempty and (ii) for every sequence $x_n \to x$ and every y_n such that $y_n \in \Gamma(x_n)$, there exists a convergent subsequence of y_n with limit $y \in \Gamma(x)$

Roughly speaking, Γ is uhc if it avoids 'imploding in the limit' at x and is not uhc if it 'jumps down' at x

Γ is *continuous* at $x \in X$ if it is both lhc *and* uhc at x

If Γ is single-valued and uhc it is continuous in the usual sense