Macroeconomics

Lecture 4: dynamic programming methods, part two

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This class

- Rough sketch of some important mathematical background
  - metric spaces and normed vector spaces
  - contraction mapping theorem
  - Blackwell's sufficient conditions for a contraction

Motivation

- To solve dynamic programming problems, want to find fixed points

$$v = Tv$$

where T is an operator like

$$Tv(k) \equiv \max_{0 \leq x \leq f(k)} \left[u(f(k) - x) + \beta v(x) \right]$$

- We will approach this problem by viewing functions v as elements in an abstract vector space and then studying the convergence properties of sequences v_n of such functions induced by T

$$v_{n+1}(k) = Tv_n(k)$$

Vector spaces

A *vector space* over $\mathbb{R}$ is a nonempty set X together with the operations of vector addition and scalar multiplication such that for all $x, y, z \in X$ and any $a, b \in \mathbb{R}$

(i)	$(x + y) + z = x + (y + z)$	(associativity of addition)
(ii)	$x + y = y + x$	(commutativity of addition)
(iii)	there exists unique $\mathbf{0} \in X$ such that $x + \mathbf{0} = x$	
(iv)	there exists unique $-x \in X$ such that $x + (-x) = \mathbf{0}$	
(v)	$a(bx) = (ab)x$	(compatibility)
(vi)	$1x = x$	(identity element)
(vii)	$a(x + y) = ax + ay$	(distributivity wrt vector addition)
(viii)	$(a + b)x = ax + bx$	(distributivity wrt scalar addition)

Remark: there is nothing here that says a vector need be *finite*.

Examples

These are vector spaces:

- any finite-dimensional Euclidean space $\mathbb{R}^n$
- the set of infinite sequences $\{x_0, x_1, \dots\}$ with each $x_i \in \mathbb{R}$
- the set of continuous functions on the interval $[a, b]$

These are *not* vector spaces:

- the unit circle in $\mathbb{R}^2$
- the set of integers $\mathbb{Z} = \{\dots, -1, 0, +1, \dots\}$
- the set of nonnegative functions on $[a, b]$

We will view functions v as elements in a suitably chosen vector space.

Will be interested in whether a sequence v_n of such functions *converges*.

This requires a notion of the *distance* between such functions.

Metric spaces

A *metric space* is a nonempty set S and a metric (distance function) $d : S \times S \to \mathbb{R}$ such that for all $x, y, z \in S$

(i) $d(x, y) = 0$ iff $x = y$

(ii) $d(x, y) = d(y, x)$ (symmetry)

(iii) $d(x, y) \leq d(x, z) + d(z, y)$ (triangle inequality)

REMARK: the set S here need not be a vector space.

Examples

These are metric spaces:

- any nonempty set S with the *discrete metric*, $d(x, y) = \mathbb{1}\{x \neq y\}$
- any finite-dimensional $\mathbb{R}^n$ with a distance of the form

$$d(x, y) = \left(\sum_{i=1}^{n} |x_i - y_i|^p \right)^{1/p}, \qquad 1 \leq p < \infty$$

and

$$d(x, y) = \max_i \left(|x_i - y_i| \right), \qquad p = \infty$$

- the set of continuous functions on $[a, b]$ with a distance of the form

$$d(x, y) = \left(\int_a^b |x(t) - y(t)|^p \, dt \right)^{1/p}, \qquad 1 \leq p < \infty$$

and

$$d(x, y) = \max_{a \leq t \leq b} \left(|x(t) - y(t)| \right), \qquad p = \infty$$

Normed vector spaces

A *normed vector space* is a vector space S and a norm $\|\cdot\|: S \to \mathbb{R}$ such that for all $x, y \in S$ and any $a \in \mathbb{R}$

(i) $\|x\| = 0$ iff $x = \mathbf{0}$

(ii) $\|ax\| = |a| \cdot \|x\|$

(iii) $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality)

Remark: any normed vector space can be viewed as a metric space by taking $d(x, y) = \|x - y\|$.

Examples

These are normed vector spaces:

- any finite-dimensional $\mathbb{R}^n$ with

$$\|x\| = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}, \qquad 1 \leq p < \infty$$

  and

$$\|x\| = \max_i \left(\, |x_i| \, \right)$$

- the set of continuous functions on $[a, b]$ with

$$\|x\| = \left(\int_a^b |x(t)|^p \, dt \right)^{1/p}, \qquad 1 \leq p < \infty$$

  and

$$\|x\| = \max_{a \leq t \leq b} \left(\, |x(t)| \, \right)$$

Convergence

A sequence $\{x_n\}_{n=0}^{\infty}$ *converges* to $x \in S$ if for any $\varepsilon > 0$ there exists an N_ε such that

$$d(x_n, x) < \varepsilon \qquad \text{for all } n \geq N_\varepsilon$$

REMARK: in other words, the sequence $\{x_n\}$ convergences to $x \in S$ if the sequence of real numbers $\{d(x_n, x)\}$ converges to zero.

Cauchy criterion

A sequence $\{x_n\}_{n=0}^{\infty}$ satisfies the *Cauchy criterion* if for any $\varepsilon > 0$ there exists an N_ε such that

$$d(x_n, x_m) < \varepsilon \qquad \text{for all } n, m \geq N_\varepsilon$$

REMARK: this is a weaker notion than convergence, but can be checked without knowledge of a candidate limit x.

Examples

The sequence $\{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ is a Cauchy sequence in $\mathbb{R}$.

The sequence $\{+1, -1, +1, -1, \ldots\}$ is *not* a Cauchy sequence in $\mathbb{R}$.

Cauchy sequences

Every convergent sequence is Cauchy.

Every Cauchy sequence is bounded.

But not every Cauchy sequence converges.

Completeness

A metric space (S, d) is *complete* if every Cauchy sequence in S converges to a point in S.

A normed vector space that is complete is known as a *Banach space*.

Examples

These are complete normed vector spaces:

- any finite-dimensional $\mathbb{R}^n$ with

$$\|x\| = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}, \qquad 1 \le p < \infty$$

or

$$\|x\| = \max_i \left(|x_i| \right)$$

- the set of continuous functions on $[a, b]$ with

$$\|x\| = \max_{a \le t \le b} \left(|x(t)| \right)$$

This is *not* a complete normed vector space:

- the set of continuous functions on $[a, b]$ with

$$\|x\| = \left(\int_a^b |x(t)|^p \, dt \right)^{1/p}, \qquad 1 \le p < \infty$$

An important space

Let $X \subseteq \mathbb{R}^n$ and let $C(X)$ be the set of bounded continuous functions $f : X \to \mathbb{R}$ equipped with the norm

$$\|f\| = \sup_{x \in X} \left(|f(x)| \right)$$

Then $C(X)$ is complete.

Remark: if X is compact, then every continuous $f : X \to \mathbb{R}$ is bounded. Otherwise the restriction to bounded functions is needed.

This is known as the *sup norm* or *uniform norm*.

Contraction mappings

Let (S, d) be a metric space and let $T : S \to S$. Then T is a *contraction mapping* (with modulus β) if for some $\beta \in (0, 1)$

$$d(Tx, Ty) \leq \beta d(x, y), \qquad \text{for all } x, y \in S$$

REMARK: in other words, applying T brings x and y closer together.

Example

Let $S = [a, b]$ and $d(x, y) = |x - y|$. Then T is a contraction mapping if for some $\beta \in (0, 1)$

$$\frac{|Tx - Ty|}{|x - y|} \leq \beta < 1, \qquad \text{for all } x, y \in S \text{ with } x \neq y$$

In other words, if slope of T is uniformly less than one in absolute value.

Contraction mapping theorem

Let (S, d) be a complete metric space and let $T : S \to S$ be a contraction mapping. Then T has a *unique* fixed point $x = Tx$ in S.

This is sometimes known as the *Banach fixed point theorem.*

Proof (sketch)

Step (i). *Showing that iterates of T form a Cauchy sequence.*

Fix any initial $x_0 \in S$ and generate $x_n = T^n x_0$ for $n = 0, 1, \ldots$ via the recursion $x_{n+1} = Tx_n$. Since T is a contraction

$$d(x_2, x_1) = d(Tx_1, Tx_0) \leq \beta d(x_1, x_0)$$

Similarly for any $n \geq 1$

$$d(x_{n+1}, x_n) \leq \beta^n d(x_1, x_0)$$

Hence for any $m > n$, by repeated use of the triangle inequality

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m-1}) + \cdots + d(x_{n+2}, x_{n+1}) + d(x_{n+1}, x_n) \\ &\leq [\beta^{m-1} + \cdots + \beta^{n+1} + \beta^n] d(x_1, x_0) \\ &= \beta^n [\beta^{m-n-1} + \cdots + \beta + 1] d(x_1, x_0) \\ &\leq \frac{\beta^n}{1 - \beta} d(x_1, x_0) \end{aligned}$$

Proof (sketch)

Hence $\{x_n\}$ is Cauchy. Since S is complete, $x_n \to x$ in S.

Step (ii). *Showing that limit is a fixed point of* T*.*

Again by the triangle inequality,

$$d(Tx, x) \leq d(Tx, T^n x_0) + d(T^n x_0, x)$$

$$\leq \beta d(x, T^{n-1} x_0) + d(T^n x_0, x)$$

In the limit as $n \to \infty$ the RHS terms both go to zero.

Hence in the limit $d(Tx, x) = 0$ or $x = Tx$.

Proof (sketch)

Step (iii). *Showing that this fixed point is unique.*

Suppose not. Then there exists another fixed point $x' \neq x$ such that

$$0 < \delta \equiv d(x, x') = d(Tx, Tx') \leq \beta d(x, x') = \beta\delta$$

But this is impossible since $\beta \in (0, 1)$. Hence the fixed point is unique.

Blackwell's sufficient conditions

Let $X \subseteq \mathbb{R}^n$ and let $B(X)$ be the set of bounded functions $f : X \to \mathbb{R}$ equipped with the sup norm. Let T be a mapping from $B(X)$ to $B(X)$ satisfying:

(i) (monotonicity) $f \leq g$ implies $Tf \leq Tg$ for all $f, g \in B(X)$

(ii) (discounting) there exists $\beta \in (0, 1)$ such that

$$T(f + a) \leq Tf + \beta a, \qquad \text{for all } f \in B(X)$$

where $f + a$ is the function $f(x) + a$ for any $x \in X$ and any $a \geq 0$

Then T is a contraction mapping.

REMARK: these are sufficient conditions only, they are not necessary. In applications, they are often quite easy to check.

Proof (sketch)

Let $v, w \in B(X)$. Then

$$v(x) = v(x) + w(x) - w(x)$$

$$\leq w(x) + |v(x) - w(x)|$$

$$\leq w(x) + \|v - w\|, \qquad \text{for all } x \in X$$

Since T satisfies (i) and (ii) we have

$$Tv \leq T(w + \|v - w\|) \leq Tw + \beta\|v - w\|$$

Repeating this argument with the roles of v, w reversed

$$Tw \leq T(v + \|v - w\|) \leq Tv + \beta\|v - w\|$$

Proof (sketch)

These last two inequalities imply

$$|Tv(x) - Tw(x)| \leq \beta \|v - w\|, \qquad \text{for all } x \in X$$

Hence indeed

$$\|Tv - Tw\| \leq \beta \|v - w\|$$

So T is a contraction.

Application to the growth model

Consider our usual Bellman operator:

$$Tv(k) \equiv \max_x \left[u(f(k) - x) + \beta v(x) \right]$$

(i) *Monotonicity.* If $v \leq w$ then

$$u(f(k) - x) + \beta v(x) \leq u(f(k) - x) + \beta w(x), \qquad \text{for all } x$$

so

$$\max_x \left[u(f(k) - x) + \beta v(x) \right] \leq \max_x \left[u(f(k) - x) + \beta w(x) \right]$$

so

$$Tv(k) \leq Tw(k)$$

Hence T satisfies the monotonicity property.

Application to the growth model

(ii) *Discounting.* For any $a \geq 0$ we have

$$T(v+a)(k) = \max_x \left[u(f(k)-x) + \beta(v(x)+a) \right]$$

$$= \max_x \left[u(f(k)-x) + \beta v(x) \right] + \beta a$$

$$= Tv(k) + \beta a$$

Hence T satisfies the discounting property.

Since (i) and (ii) are satisfied, T is a contraction.

Next class

- This argument takes as granted that $v \in C(X)$ so that we may apply the contraction mapping theorem

- What guarantees that $v \in C(X)$?

- What else can we say about the value function? Is it increasing? concave? differentiable?