Macroeconomics

Lecture 3: dynamic programming methods, part one

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1st Semester 2019

This class

- Introduction to deterministic dynamic programming
  - recursive approach to the growth model
  - key concepts: value function, Bellman equation etc

Sequence problem

- We are now familiar with the following *sequence problem*

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \, u(c_t), \qquad 0 < \beta < 1$$

subject to the sequence of constraints,

$$c_t \geq 0, \qquad \text{and} \qquad c_t + k_{t+1} \leq f(k_t) + (1-\delta)k_t$$

with the initial condition

$$k_0 > 0 \qquad \text{given}$$

- Unless stated otherwise, assume $u(c)$ and $f(k)$ strictly increasing and strictly concave

- To streamline notation, let $\delta = 1$ (equivalently, let $f(k_t)$ denote total supply of goods at beginning of t) so that $c_t = f(k_t) - k_{t+1}$

Value function

- Let $v(k_0)$ denote the maximized objective function

$$v(k_0) \equiv \max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(f(k_t) - k_{t+1}), \qquad 0 < \beta < 1$$

- This is known as a *value function*

- It is the value to the planner of being endowed with $k_0 > 0$ and then proceeding optimally

- Key to making this operationally useful is that the value function has a simple *recursive* structure

Value function

- To see this recursive structure, let's break the sum up

$$v(k_0) \equiv \max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \, u(f(k_t) - k_{t+1})$$

$$= \max_{\{k_{t+1}\}_{t=0}^{\infty}} \left[u(f(k_0) - k_1) + \beta \sum_{t=1}^{\infty} \beta^{t-1} \, u(f(k_t) - k_{t+1}) \right]$$

$$= \max_{k_1} \left[u(f(k_0) - k_1) + \beta \max_{\{k_{t+1}\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \beta^{t-1} \, u(f(k_t) - k_{t+1}) \right]$$

$$= \max_{k_1} \left[u(f(k_0) - k_1) + \beta v(k_1) \right]$$

Notational conventions

- We do not know the value function but we know it satisfies

$$v(k_0) = \max_{k_1} \left[u(f(k_0) - k_1) + \beta v(k_1) \right]$$

- Nothing special about periods $t = 0$ and $t = 1$, we could write

$$v(k_t) = \max_{k_{t+1}} \left[u(f(k_t) - k_{t+1}) + \beta v(k_{t+1}) \right]$$

- But for this *stationary* problem, there is nothing special about the time periods at all, we may as well write

$$v(k) = \max_{x} \left[u(f(k) - x) + \beta v(x) \right]$$

where x is just a dummy variable indexing possible choices of next period's capital stock

- Current capital stock k is the sole *state variable*, a sufficient statistic for past decisions

Bellman equation

- A recursive representation like

$$v(k) = \max_x \left[\, u(f(k) - x) + \beta v(x) \,\right]$$

is often called a *Bellman equation*, after Richard Bellman

- More generally, this is an example of a *functional equation* to be solved for an unknown function, i.e., given exogenous $u(c)$, $f(k)$ and β we need to solve for an endogenous function $v(k)$

- Much of economics (in macro and micro) involves solving problems like this. We will spend the next few weeks learning how to do so

First order condition

- Consider the RHS of the Bellman equation. Treating $v(x)$ as known, this is just like a two-period problem

$$\max_x \left[\, u(f(k) - x) + \beta v(x) \,\right]$$

- And implies the first order condition

$$u'(f(k) - x) = \beta v'(x)$$

which we could imagine solving for some $x = g(k)$. But we don't know $v(x)$ and hence don't know $v'(x)$

Aside on parameterized optimization problems

- Suppose we seek to maximize $u(x, \theta)$ by choice of x given parameter θ

$$v(\theta) \equiv \max_x u(x, \theta)$$

- Let $x = g(\theta)$ achieve the maximum

$$g(\theta) \equiv \operatorname*{argmax}_x u(x, \theta)$$

so that

$$v(\theta) = u(g(\theta), \theta)$$

- What do we know about $v(\theta)$? What about $v'(\theta)$?

Envelope theorem

- Suppose further that $x = g(\theta)$ is characterized by the first order condition

$$\frac{\partial u(g(\theta), \theta)}{\partial x} = 0$$

- Then we have the *envelope theorem*

$$v'(\theta) = \frac{\partial u(g(\theta), \theta)}{\partial x} g'(\theta) + \frac{\partial u(g(\theta), \theta)}{\partial \theta}$$

$$= \frac{\partial u(g(\theta), \theta)}{\partial \theta}$$

- The total derivative of the value function with respect to θ is given by the partial derivative of the objective function with respect to θ evaluated at the optimum

Envelope condition

- Let's apply this to our dynamic programming problem

$$v(k) = \max_x \left[u(f(k) - x) + \beta v(x) \right]$$

- Therefore

$$v'(k) = \frac{\partial}{\partial k}\left[u(f(k) - x) + \beta v(x) \right], \qquad x = g(k)$$

$$= u'(f(k) - x) f'(k), \qquad x = g(k)$$

$$= u'(f(k) - g(k)) f'(k)$$

Policy function

- In dynamic programming problems like this, the function

$$g(k) \equiv \underset{x}{\text{argmax}} \left[u(f(k) - x) + \beta v(x) \right]$$

is known as the *policy function* or *decision rule*

- Iterating on the policy function gives the sequence of capital stocks

$$\begin{aligned} k_1 &= g(k_0) \\ k_2 &= g(k_1) = g(g(k_0)) \\ &\vdots \\ k_{t+1} &= g(k_t) = g^t(k_0) \end{aligned}$$

- Properties of $g(k)$ determine the properties of the optimal sequence of k_t. Steady states satisfy $k^* = g(k^*)$. A steady state is locally stable if $|g'(k^*)| < 1$, and so on

Policy function

- With the policy function $g(k)$, can then recover consumption

$$c(k) = f(k) - g(k)$$

- To relate this back to our usual saddle-path phase diagram, this $c(k)$ is the *stable-arm* of the saddle-path

- Notice that this $c(k)$ is the same as

$$c(k) \equiv \underset{c}{\operatorname{argmax}} \left[u(c) + \beta v(f(k) - c) \right]$$

Combining first order and envelope conditions

- To summarize, we have the problem

$$v(k) = \max_x \left[u(f(k) - x) + \beta v(x) \right]$$

- The first order condition for this problem is

$$u'(f(k) - g(k)) = \beta v'(g(k))$$

- The envelope condition says

$$v'(k) = u'(f(k) - g(k))f'(k)$$

- Evaluating this at $g(k)$ gives the somewhat cumbersome

$$v'(g(k)) = u'(f(g(k)) - g(g(k)))f'(g(k))$$

Euler equation

- Hence we can write the *Euler equation*

$$u'(f(k) - g(k)) = \beta u'(f(g(k)) - g(g(k)))f'(g(k))$$

- This Euler equation can also be viewed as a functional equation, to be solved for the policy function $g(k)$

- Using $k_{t+1} = g(k_t)$ and $k_{t+2} = g(g(k_t))$ etc, in sequence notation this is just the usual condition

$$u'(f(k_t) - k_{t+1}) = \beta u'(f(k_{t+1}) - k_{t+2})f'(k_{t+1})$$

Dynamical systems compared

- Policy function from the dynamic programming problem

$$k_{t+1} = g(k_t), \qquad k_0 > 0 \text{ given}$$

This is a *one-dimensional* dynamical system in k_t

- From the Euler equation

$$u'(f(k_t) - k_{t+1}) = \beta u'(f(k_{t+1}) - k_{t+2}) f'(k_{t+1}), \qquad k_0 > 0 \text{ given}$$

This is a *two-dimensional* dynamical system in k_t

- Why the difference?

Bellman equations vs. Euler equations

- For a given problem, can either

  (i) attempt to solve for value function $v(k)$ from Bellman equation and then determine policy function $g(k)$, or

  (ii) attempt to solve for policy function $g(k)$ from Euler equation

- Solving Euler equations generally faster than solving Bellman equations, so when problem is 'well-behaved' (ii) is often preferable

- Solving Bellman equations, while slower, is generally more robust

Method of successive approximations

- Suppose we had some candidate value function $v_0(k)$
- Define a new value function by

$$v_1(k) = \max_x \left[u(f(k) - x) + \beta v_0(x) \right]$$

and test whether $v_1(k)$ equals $v_0(k)$ or not

- Unless we are lucky, $v_1(k) \neq v_0(k)$. But suppose we keep iterating

$$v_{n+1}(k) = \max_x \left[u(f(k) - x) + \beta v_n(x) \right], \qquad n = 0, 1, \ldots$$

- What happens to the sequence of functions v_n as $n \to \infty$?

A suboptimal policy

- Suppose we followed any *feasible* policy $g_0(k)$

- Let $v_0(k)$ be the value of this generally suboptimal policy

$$v_0(k_0) = \sum_{t=0}^{\infty} \beta^t \, u(f(k_t) - g_0(k_t))$$

such that, for arbitrary k,

$$v_0(k) = u(f(k) - g_0(k)) + \beta v_0(g_0(k))$$

A suboptimal policy

- Then we have

$$v_1(k) = \max_x \left[u(f(k) - x) + \beta v_0(x) \right]$$

$$\geq \left[u(f(k) - g_0(k)) + \beta v_0(g_0(k)) \right]$$

$$= v_0(k)$$

(can do no worse by choosing optimally today)

A suboptimal policy

- Likewise

$$
\begin{aligned}
v_2(k) &= \max_x \left[\, u(f(k) - x) + \beta v_1(x) \,\right] \\
&\geq \max_x \left[\, u(f(k) - x) + \beta v_0(x) \,\right] \\
&= v_1(k)
\end{aligned}
$$

- Continuing in this way, $v_{n+1}(k) \geq v_n(k)$ for $n = 0, 1, \ldots$

Bellman operator

- Let Tv denote the function created by the RHS of the Bellman equation

$$Tv(k) \equiv \max_x \left[u(f(k) - x) + \beta v(x) \right]$$

- T is an *operator* that takes as an input a function v and returns a new function Tv

- In this notation

$$Tv_n(k) \equiv \max_x \left[u(f(k) - x) + \beta v_n(x) \right]$$

- Can iterate on Bellman operator to get $v_{n+1} = Tv_n$ for $n = 0, 1, \ldots$

Iterating on Bellman operator

- Notice that solving the Bellman equation is equivalent to solving the fixed point problem

$$v = Tv$$

- We have seen that

$$v_{n+1} = Tv_n \geq v_n$$

- Does this increasing sequence v_n converge to a limit v as $n \to \infty$?

Next class

- Sketch of mathematical background
  - contraction mapping theorem
  - Blackwell's sufficient conditions (for a contraction)