Macroeconomics

Lecture 2: review of neoclassical growth model

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This class

- Review of neoclassical growth model
  - production, not an endowment economy
  - endogenous return on capital
  - links capital accumulation with consumption/savings decisions

Neoclassical growth model

- Discrete time $t = 0, 1, 2, \ldots$

- Output Y_t is produced with physical capital K_t and labor L_t according to the *aggregate production function*

  $$Y_t = F(K_t, A_t L_t)$$

  with *labor-augmenting* productivity A_t

- Physical capital depreciates at rate δ

  $$K_{t+1} = (1-\delta)K_t + I_t, \qquad 0 < \delta < 1, \qquad K_0 > 0$$

- Goods may be either consumed or invested

  $$C_t + I_t = Y_t$$

- Gives the sequence of *resource constraints*, one for each date

  $$C_t + K_{t+1} = F(K_t, L_t) + (1-\delta)K_t, \qquad K_0 > 0$$

Aggregate production function

- Each input has positive marginal product

$$F_K(K, L) > 0, \qquad F_L(K, L) > 0$$

- Each input suffers from diminishing returns

$$F_{KK}(K, L) < 0, \qquad F_{LL}(K, L) < 0$$

- *Constant returns to scale*, i.e., if both inputs scaled by common factor $c > 0$ then

$$F(cK, cL) = cF(K, L)$$

- Some analysis is simplified by assuming the '*Inada conditions*'

$$F_K(0, L) = F_L(K, 0) = \infty,$$
$$F_K(\infty, L) = F_L(K, \infty) = 0$$

and that both inputs are essential, i.e., $F(0, L) = F(K, 0) = 0$

Intensive form

- In efficiency units

$$y \equiv \frac{Y}{AL}, \qquad k \equiv \frac{K}{AL}, \ldots \qquad \text{etc}$$

- Using constant returns to scale

$$y = \frac{Y}{AL} = \frac{F(K, AL)}{AL} = F(\frac{K}{AL}, 1) = F(k, 1) \equiv f(k)$$

- Intensive version of the production function

$$y = f(k), \qquad f'(k) > 0, \qquad f''(k) < 0,$$

and with Inada conditions

$$f'(0) = \infty, \qquad f'(\infty) = 0$$

Intensive form

- To streamline notation, suppose constant $L_t = L$ and $A_t = A$

- Then intensive form of the resource constraint is simply

$$c_t + k_{t+1} = f(k_t) + (1-\delta)k_t, \qquad k_0 > 0$$

Social planner's problem

- Social planner chooses stream $c_t \geq 0$ to maximize

$$\sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to sequence of resource constraints

$$c_t + k_{t+1} = f(k_t) + (1 - \delta)k_t, \qquad k_0 > 0$$

- Infinite horizon keeps model 'stationary', no life-cycle effects

- Can be decentralized, focus on planner's problem for simplicity

Social planner's problem

- Lagrangian with multiplier $\lambda_t \geq 0$ for each resource constraint

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(c_t) + \sum_{t=0}^{\infty} \lambda_t \big[f(k_t) + (1-\delta)k_t - c_t - k_{t+1} \big]$$

- Key first order conditions

$$c_t : \qquad \beta^t u'(c_t) - \lambda_t = 0$$

$$k_{t+1} : \qquad -\lambda_t + \lambda_{t+1} \big[f'(k_{t+1}) + 1 - \delta \big] = 0$$

$$\lambda_t : \qquad f(k_t) + (1-\delta)k_t - c_t - k_{t+1} = 0$$

These hold at every date

Consumption Euler equation

- Eliminating the Lagrange multipliers

$$u'(c_t) = \beta u'(c_{t+1})\big[f'(k_{t+1}) + 1 - \delta\big]$$

- Same as last class if we recognize that the 'return on capital' is

$$\text{`}\, R_{t+1} = f'(k_{t+1}) + 1 - \delta \,\text{'}$$

- Planner equates marginal rate of substitution (MRS) between t and $t+1$ with marginal rate of transformation (MRT)

  – MRS between t and $t+1$

  $$\frac{u'(c_t)}{\beta u'(c_{t+1})}$$

  – MRT between t and $t+1$

  $$f'(k_{t+1}) + 1 - \delta$$

Dynamical system

- Gives a system of two nonlinear difference equations in c_t, k_t

$$u'(c_t) = \beta u'(c_{t+1})\big[f'(k_{t+1}) + 1 - \delta\big]$$

  and

$$c_t + k_{t+1} = f(k_t) + (1-\delta)k_t$$

- Two boundary conditions: (i) initial $k_0 > 0$ given, and (ii) the '*transversality condition*'

$$\lim_{T\to\infty} \beta^T u'(c_T)k_{T+1} = 0$$

  (analogous to $k_{T+1} = 0$ we would have in finite-horizon model)

Steady state

- Steady state where $\Delta c_t = 0$ and $\Delta k_t = 0$. Let c^*, k^* denote steady state values. These are determined by

$$1 = \beta \left[f'(k^*) + 1 - \delta \right]$$

and

$$c^* + k^* = f(k^*) + (1 - \delta) k^*$$

- Steady state Euler equation pins down k^*, resource constraint then determines c^*, in particular

$$c^* = f(k^*) - \delta k^*$$

Modified golden rule

- Let $C(k)$ denote consumption sustained by holding k_t fixed at k

$$C(k) \equiv f(k) - \delta k$$

- $C(k)$ is maximized at the '*golden rule*' level, where

$$f'(k) = \delta$$

- Steady state capital stock determined by

$$f'(k) = \rho + \delta, \qquad \rho \equiv \frac{1}{\beta} - 1 > 0$$

where $\rho > 0$ is the pure *rate of time preference*

- Hence steady state capital is less than the golden rule level

Qualitative dynamics

- Consumption dynamics

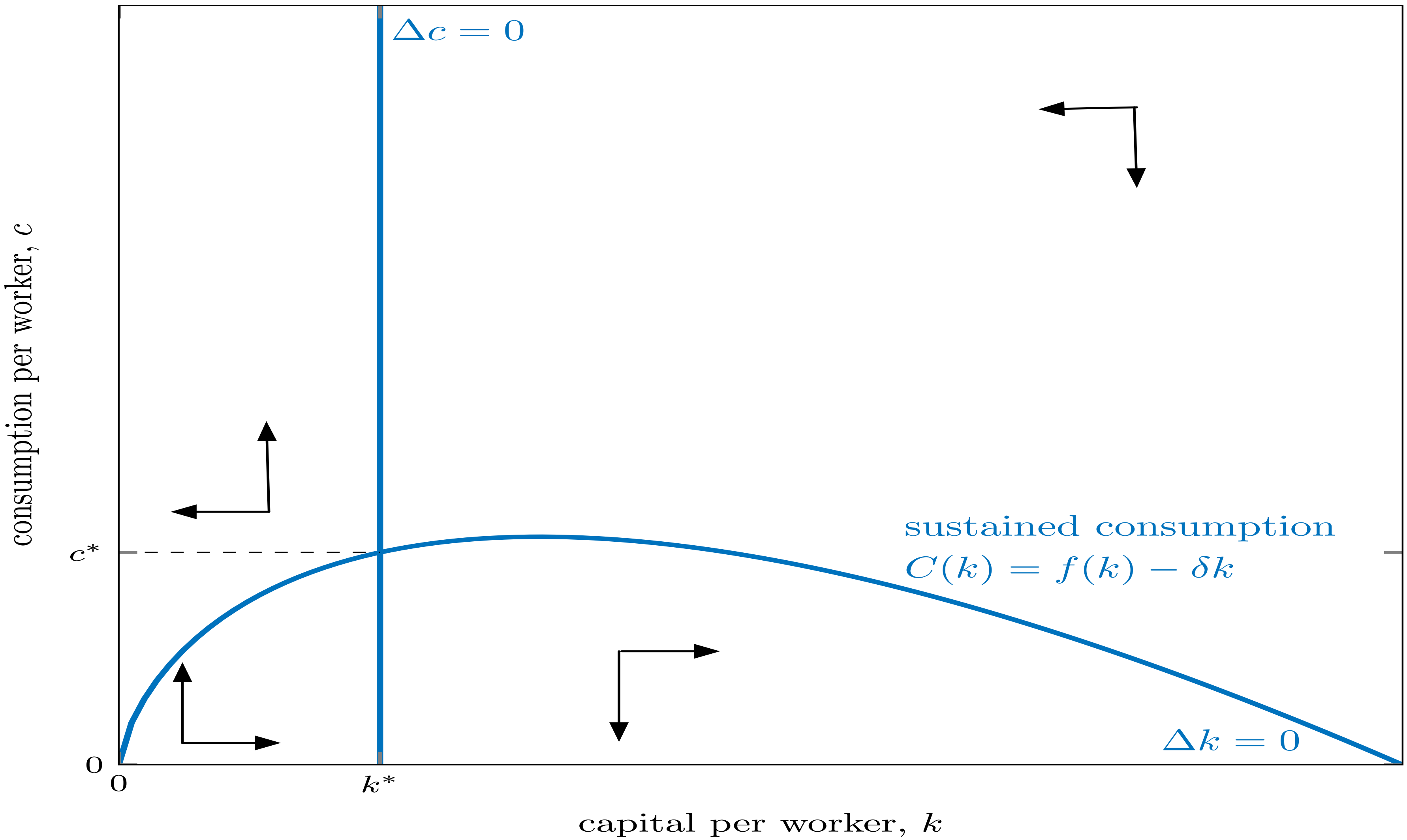
$$c_{t+1} > c_t \qquad \Leftrightarrow \qquad k_{t+1} < k^*$$

- Capital dynamics

$$k_{t+1} > k_t \qquad \Leftrightarrow \qquad c_t < C(k_t)$$

- Divides k_t, c_t space into *four regions*. Flows can be analyzed with a two-dimensional phase diagram

Phase diagram in k_t, c_t space



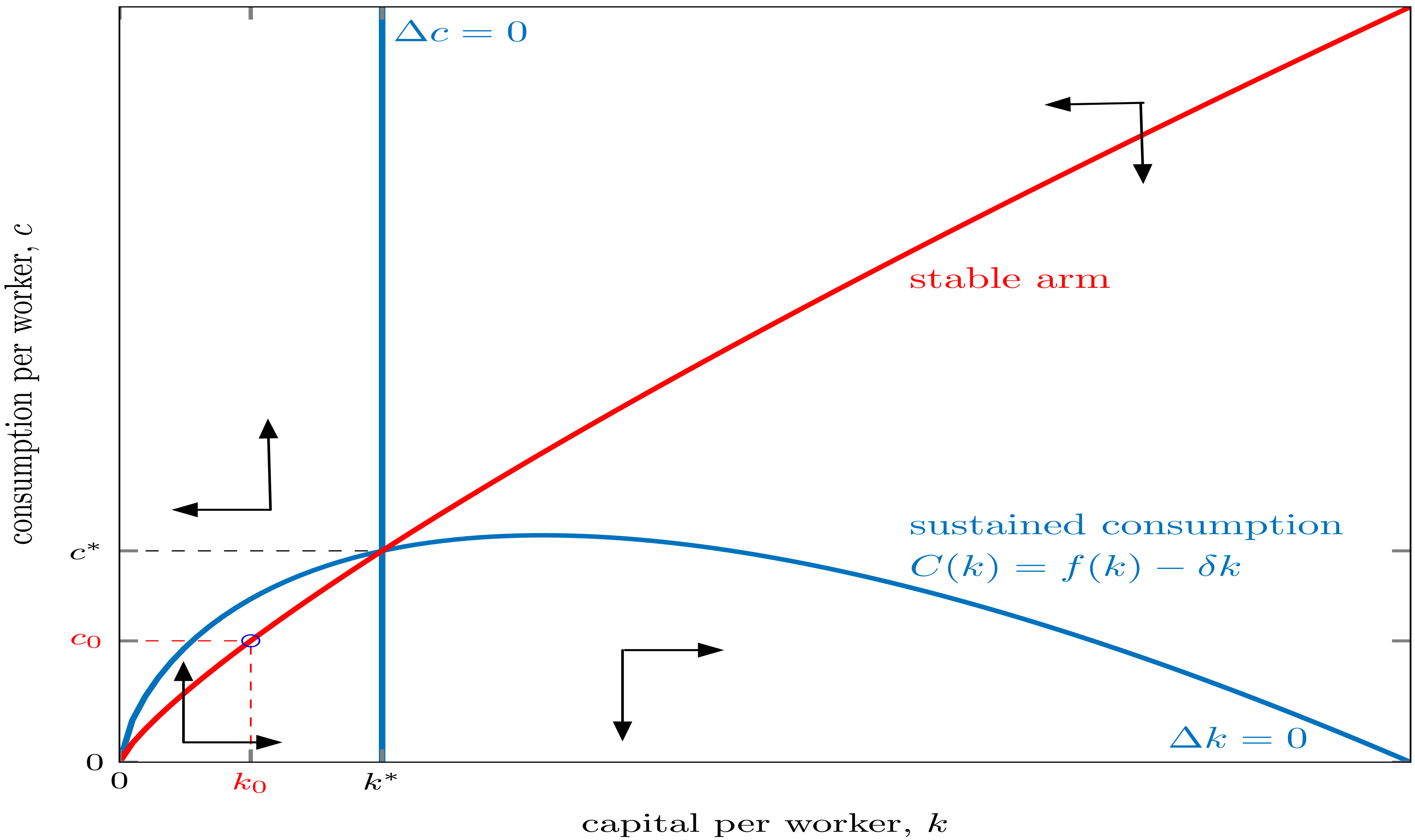
Determining c_0

- Capital k_0 is *pre-determined* (historically given) at date $t = 0$
- Consumption c_0 not pre-determined, can '*jump*' within feasible set

$$0 \leq c_0 \leq C(k_0) + k_0$$

- Consumption c_0 jumps to '*stable arm*' of the dynamical system
- Initial consumption is the one *degree of freedom* that can be used to avoid undesirable trajectories

Stable arm



$\Delta c = 0$
stable arm
sustained consumption
$C(k) = f(k) - \delta k$
$\Delta k = 0$
consumption per worker, c
c^*
c_0
0
0
k_0
k^*
capital per worker, k

Local dynamics

- Let $\hat{x}_t$ denote the *log-deviation* of x_t from its steady state value

$$\hat{x}_t \equiv \log\left(\frac{x_t}{x^*}\right) \approx \frac{x_t - x^*}{x^*}$$

- Can show that local to steady state c^*, k^* dynamics given by

$$\begin{pmatrix} \hat{c}_{t+1} \\ \\ \hat{k}_{t+1} \end{pmatrix} = \begin{pmatrix} 1 - \frac{\beta f''(k^*)c^*}{\sigma(c^*)} & \frac{f''(k^*)k^*}{\sigma(c^*)} \\ \\ -\frac{c^*}{k^*} & \frac{1}{\beta} \end{pmatrix} \begin{pmatrix} \hat{c}_t \\ \\ \hat{k}_t \end{pmatrix}$$

where $\frac{1}{\sigma(c)}$ is the intertemporal elasticity of substitution

$$\frac{1}{\sigma(c)} = -\frac{u'(c)}{u''(c)c} > 0$$

- This coefficient matrix has one stable and one unstable root

Local dynamics

- Solution to this system has the form

$$\hat{k}_{t+1} = \psi_{kk}\,\hat{k}_t, \qquad \text{and} \qquad \hat{c}_t = \psi_{ck}\,\hat{k}_t$$

  where ψ_{kk} is the stable root of the coefficient matrix and where ψ_{ck} is the slope of the stable arm

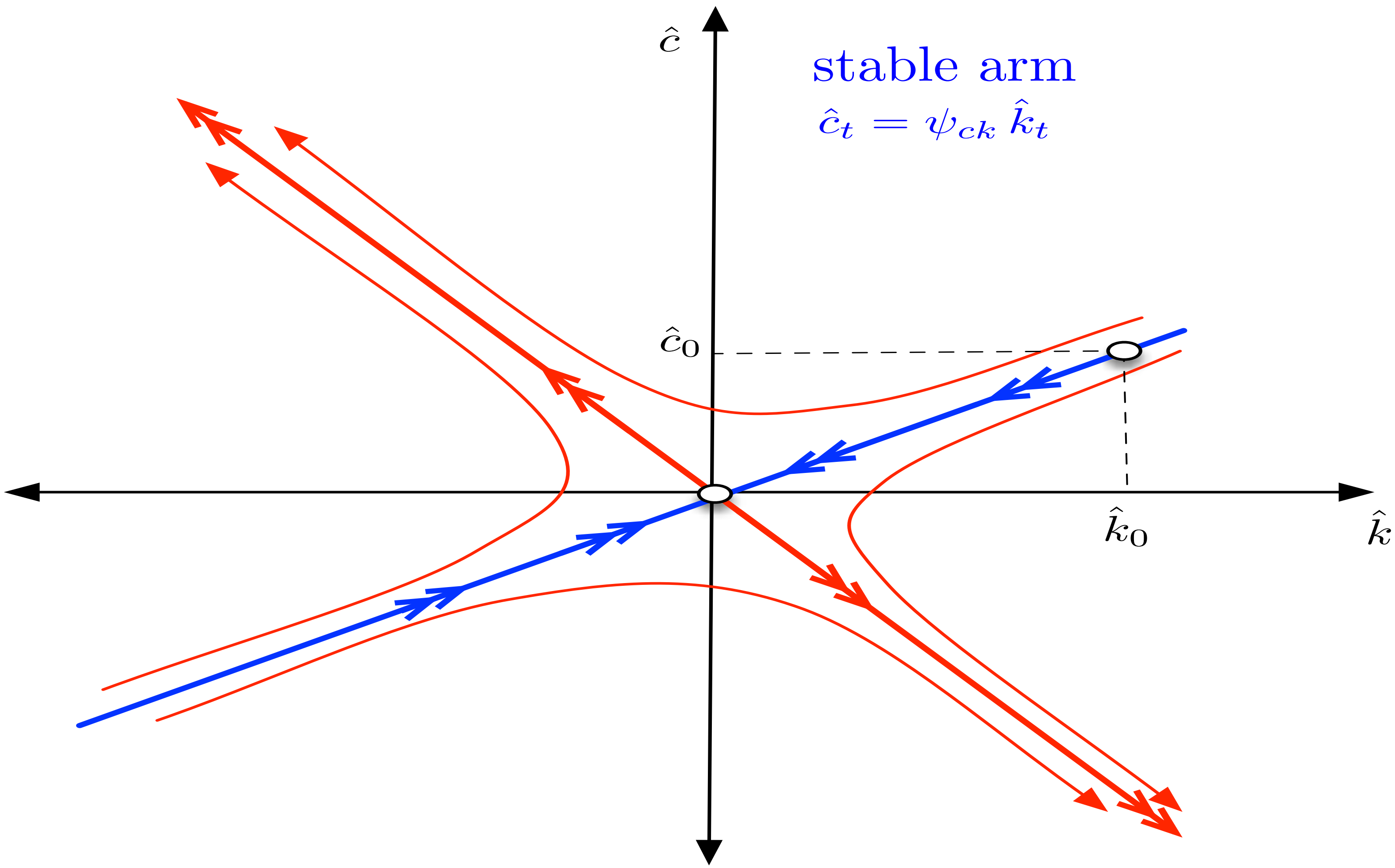
- In particular, $\psi_{kk} \in (0,1)$ is the stable root of the quadratic

$$\psi_{kk}^2 - \left(1 - \frac{\beta f''(k^*)c^*}{\sigma(c^*)} + \frac{1}{\beta}\right)\psi_{kk} + \frac{1}{\beta} = 0$$

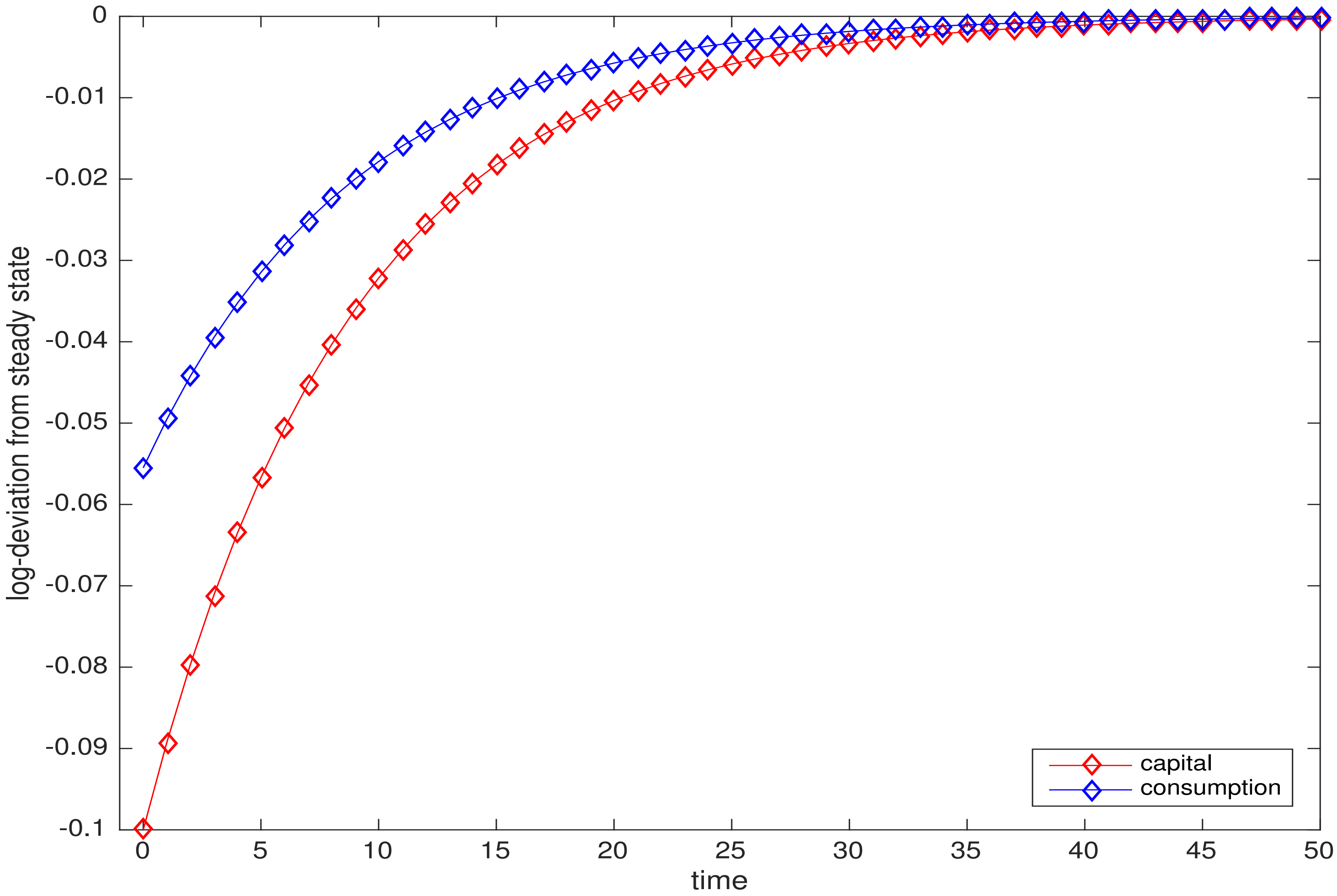
  and then

$$\psi_{ck} = \left(\frac{1}{\beta} - \psi_{kk}\right)\frac{k^*}{c^*} > 0$$

Local dynamics



Example: transition to steady state



Initial capital $\hat{k}_0 = -0.1$ (i.e., 10% below steady state). Capital $\hat{k}_{t+1} = \psi_{kk}\hat{k}_t$ and consumption $\hat{c}_t = \psi_{ck}\hat{k}_t$ with $\psi_{kk} = 0.89$ and $\psi_{ck} = 0.56$.

Next class

- Dynamic programming methods, part one
  - recursive approach to the growth model