Macroeconomics

Lecture 19: firm dynamics, part one

Chris Edmond

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This class

- Hopenhayn (1992) model of entry, exit and long-run equilibrium
  - model exposition
  - sketch of computational procedure
  - simplified static version for intuition
  - computational details and practicalities

Hopenhayn model

- Workhorse model of industry dynamics
- Steady-state model: firms enter, grow and decline, and exit, but overall distribution of firms is unchanging
- Endogenous stationary distribution of firm-size etc, straightforward comparative statics

Key elements

- Continuum of firms, no strategic interactions
- Produce with decreasing returns to scale, so well-defined size
- Idiosyncratic risk: firm-level productivities follow a Markov process
- No aggregate risk
- Fixed cost to enter, fixed cost to operate each period

Setup

- Time $t = 0, 1, 2, ...$

- Output and input prices p and w taken as given

- Output y produced with labor n given productivity z

$$y = zF(n)$$

- Static profits

$$\pi(z) = \max_{n \geq 0} \Big[pzF(n) - wn - k \Big]$$

where $k > 0$ is per-period fixed cost of operating

- Let $n(z)$ and $y(z)$ denote the associated optimal policies

Setup

- Productivity
  - $n(z), y(z), \pi(z)$ are all strictly increasing in productivity z
  - Markov process for z with transition density $f(z' \,|\, z)$
  - entrants draw initial productivity z_0 from separate distribution $g(z)$, pay sunk cost $k_e > 0$ to do so
- Timing within a period
  - incumbents decide to stay or exit, entrants decide to enter or not
  - incumbents that stay pay k, entrants pay k_e
  - *after* paying k or k_e, operating firms learn their productivity draws

Market clearing

- Can choose either p or w as numeraire. We will choose $w = 1$
- Two key aggregate variables: (i) the price p, and (ii) the cross-sectional distribution of productivity types $\mu(z)$
- Industry supply curve, endogenous

$$Y(p) = \int y(z\,;\,p)\mu(z)\,dz$$

- Market clears when

$$Y(p) = D(p)$$

for some given demand curve $D(p)$

Incumbent's problem

- Let $v(z\,;\,p)$ denote the value of incumbency to a firm with current productivity z given price p

- Bellman equation for an incumbent firm

$$v(z\,;\,p) = \pi(z,p) + \beta \max\left[0\,,\; \int v(z'\,;\,p)\, f(z'\,|\,z)\, dz'\right]$$

- An *exit threshold* $z^*(p)$ such that firm exits if $z < z^*(p)$, solves

$$\int v(z'\,;\,p)\, f(z'\,|\,z^*)\, dz' = 0$$

(for interior cases)

Entrant's problem

- Potential entrants are ex ante identical

- Pay $k_e > 0$ to enter, initial draw from $g(z)$ if they do

- Start producing next period

- Let $m \geq 0$ denote the mass of entrants, *free entry* condition

$$\beta \int v(z\,;\,p)\,g(z)\,dz \leq k_e$$

with strict equality whenever $m > 0$

Stationary equilibrium

- A *stationary equilibrium* is a value function $v(z)$, output policy $y(z)$, cutoff productivity z^*, distribution $\mu(z)$, mass of entrants m and price p such that:

  (i) taking p as given, $v(z)$, $y(z)$ and z^* solve the dynamic programming problem for an incumbent of type z

  (ii) the free-entry condition

  $$\beta \int v(z)\, g(z)\, dz \leq k_e$$

  is satisfied, with strict equality whenever $m > 0$

  (iii) the goods market clears

  $$\int y(z)\mu(z)\, dz = D(p)$$

Stationary equilibrium

(iv) the distribution $\mu(z)$ is stationary

$$\mu(z') = \int \phi(z' \,|\, z)\, \mu(z)\, dz + mg(z')$$

where the conditional distribution $\phi(z' \,|\, z)$ is given by

$$\phi(z' \,|\, z) = f(z' \,|\, z) \times \mathbb{1}[z \geq z^*]$$

Stationary distribution

- There is exit by firms with $z < z^*$ and entry $mg(z')$

- The distribution is stationary when

$$\mu(z') = \int \phi(z' \mid z)\, \mu(z)\, dz + mg(z')$$

- Suppose we discretize to a grid with n elements. Then this is a linear system of the form

$$\boldsymbol{\mu} = \boldsymbol{\Phi}\boldsymbol{\mu} + m\boldsymbol{g}$$

where $\boldsymbol{\Phi}$ is $n \times n$, $\boldsymbol{\mu}$ and $\boldsymbol{g}$ are $n \times 1$ and m is a scalar

Computing an equilibrium (sketch)

- **Step 1.** Guess output price p^0. For this price, solve the incumbent's dynamic programming problem

$$v(z\,;\,p^0) = \pi(z\,;\,p^0) + \beta \max\left[0\,,\, \int v(z'\,;\,p^0) f(z' \,|\, z)\, dz'\right]$$

The solution of this problem also implies the optimal exit rule, i.e., the $z^*(p^0)$ that solves

$$\int v(z'\,;\,p^0) f(z' \,|\, z^*)\, dz' = 0$$

- **Step 2.** Check that this price p^0 satisfies the free-entry condition

$$\beta \int v(z', p^0) g(z')\, dz' = k_e$$

For example, if the LHS is too high, then go back to Step 2 and guess a new price $p^1 < p^0$. Continue until a price p^* is found that solves the free-entry condition

Computing an equilibrium (sketch)

- **Step 3.** Guess a measure of entrants, m^0. Given this, calculate the stationary distribution $\mu^0(z)$. This solves the linear system

$$\mu(z') = \int f(z' \,|\, z)\, \mu(z)\, dz + mg(z')$$

  Observe that the RHS depends on the price found at Step 2 via the exit threshold $z^*(p^*)$

- **Step 4.** Given this $\mu^0(z)$, calculate the total industry supply and check the market clearing condition

$$Y = \int y(z, p^*)\mu^0(z)\, dz = D(p^*)$$

  For example, if the LHS is too low, then go back to Step 3 and guess new entrants $m^1 > m^0$. Continue until a m^* is found that solves the market-clearing condition

Speeding up the last step

- The stationary distribution is linearly homogeneous in m

- In terms of the discretized system above

$$\boldsymbol{\mu} = \boldsymbol{\Phi}\boldsymbol{\mu} + m\boldsymbol{g} \qquad \Rightarrow \qquad \boldsymbol{\mu} = m(\boldsymbol{I} - \boldsymbol{\Phi})^{-1}\boldsymbol{g}$$

  where $\boldsymbol{I}$ is an identity matrix

- Two implications

  – no need to use simulations to find stationary distribution $\boldsymbol{\mu}$, just set up coefficient matrix $\boldsymbol{\Phi}$ (implied by $z^*(p^*)$) and calculate directly

  – only invert $(\boldsymbol{I} - \boldsymbol{\Phi})$ once, then just rescale by m

Comparative statics

- Increase in entry cost k_e
  - increases expected discounted profits
  - decreases exit threshold z^*
    - less selection, incumbents make more profits, more continue
    - increases average age of firms
  - decreases mass of entrants m and entry/exit rate
  - increases price p

Comparative statics

- Ambiguous implications for firm-size distribution
  - *price effect*, higher k_e increases price p

    hence incumbents increase output $y(z\,;\,p)$ and employment $n(z\,;\,p)$
  - *selection effect*, higher k_e reduces productivity threshold z^*

    hence more incumbent firms are relatively-low productivity firms

Static version for intuition

- Once-and-for-all productivity draw $z \sim g(z)$, once-and-for-all endogenous exit. Production function $F(n) = n^{\alpha}$ for $0 < \alpha < 1$

- Static profit maximization problem

$$\pi(z\,;\,p) \equiv \max_{n \geq 0} \Big[pzn^{\alpha} - n - k \Big], \qquad (w = 1 \text{ is numeraire})$$

- Implies employment, output

$$n(z\,;\,p) = (\alpha pz)^{\frac{1}{1-\alpha}}, \qquad y(z\,;\,p) = zn(z\,;\,p)^{\alpha}$$

and profits

$$\pi(z\,;\,p) = (1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}(pz)^{\frac{1}{1-\alpha}} - k$$

Exit and entry conditions

- Exit threshold z^* satisfies

$$\pi(z^* \,;\, p) = 0$$

such that firms immediately exit for all $z < z^*$

- Value of a firm given once-and-for-all choices

$$v(z\,;\,p) = \max\left[0\,,\, \sum_{t=0}^{\infty} \beta^t \pi(z\,;\,p)\right] = \max\left[0, \frac{\pi(z\,;\,p)}{1-\beta}\right]$$

- Free entry condition

$$k_e = \beta \int v(z\,;\,p^*)\, g(z)\, dz = \beta \int_{z^*}^{\infty} \frac{\pi(z\,;\,p^*)}{1-\beta}\, g(z)\, dz$$

Two conditions in two unknowns z^*, p^*

Implications for selection

- Substituting the profit function into the free entry condition

$$(1-\beta)k_e = \beta \int_{z^*}^{\infty} \left[(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}} (p^* z)^{\frac{1}{1-\alpha}} - k\right] g(z)\, dz$$

- Using the exit condition for z^* to eliminate p^* gives

$$(1-\beta)\frac{k_e}{k} = \beta \int_{z^*}^{\infty} \left[\left(\frac{z}{z^*}\right)^{\frac{1}{1-\alpha}} - 1\right] g(z)\, dz$$

- Increase in k_e (or decrease in k) reduces cutoff z^* and increases p^*, i.e., larger entry barriers *weaken the selection effect and allow more unproductive firms to operate*

Solving the Hopenhayn model

- Back to dynamic version

- Suppose productivity follows n-state Markov chain on z_i with transition probabilities f_{ij}

- Given price p, value function is a n-vector with elements $v_i(p)$, i.e,

$$v_i(p) \equiv v(z_i\,;\,p), \quad \pi_i(p) \equiv \pi(z_i\,;\,p), \quad y_i(p) \equiv y(z_i\,;\,p), \quad \text{etc}$$

- Bellman equation for incumbent firm is then

$$v_i(p) = \pi_i(p) + \beta \max\left[0\,,\,\sum_{j=1}^{n} v_j(p)\,f_{ij}\right]$$

- Solve this by value function iteration

Free entry

- Let $g_i = g(z_i)$ denote initial distribution over nodes z_i

- Given $v_i(p)$ that solves incumbent's problem, free entry condition is

$$v^e(p) \equiv \beta \sum_{i=1}^{n} v_i(p) g_i = k_e$$

  whenever there is positive entry, $m > 0$ (for some parameter values, may have $m = 0$ in which case $v^e(p) < k_e$, see below)

- Easy to show that $v^e(0) < 0$ and $v^e(p)$ monotone increasing in p, so interior solutions (with $m > 0$) can be found by bisection

- Intuitively, if $v^e(p) > k_e$ then reduce price to discourage entry but if $v^e(p) < k_e$ then increase price to encourage entry

Exit decisions

- Given p^*, we know incumbent value function $v_i(p^*)$

- Exit threshold $z^*(p^*)$ then found from

$$z^*(p^*) = z_{i^*}, \qquad i^* := \min_i \left[\sum_{j=1}^n v_j(p^*)\, f_{ij} \geq 0 \right]$$

All firms with $z_i < z^*(p^*)$ *exit*, all firms with $z_i \geq z^*(p^*)$ *continue*

- Collect exit decisions into a vector with elements

$$x_i(p^*) = 1 \qquad \text{if } z_i < z^*(p^*)$$
$$x_i(p^*) = 0 \qquad \text{if } z_i \geq z^*(p^*)$$

Stationary distribution

- Let $\mu_i = \mu(z_i)$ denote mass of firms with productivity z_i

- Vector $\boldsymbol{\mu}$ solves

$$\boldsymbol{\mu} = \boldsymbol{\Phi}(p^*)\boldsymbol{\mu} + m\boldsymbol{g}$$

where $n \times n$ coefficient matrix $\boldsymbol{\Phi}(p^*)$ has elements

$$\phi_{ij}(p^*) = (1 - x_j(p^*))\, f_{ji}, \qquad i, j = 1, ..., n$$

- Stationary distribution

$$\boldsymbol{\mu} = m(\boldsymbol{I} - \boldsymbol{\Phi}(p^*))^{-1}\boldsymbol{g} \equiv \boldsymbol{\mu}(m, p^*)$$

for some m yet to be determined

Market clearing

- Industry demand curve, exogenous $D(p)$

- Industry supply curve, endogenous

$$Y(m,p) = \sum_{i=1}^{n} y_i(p)\mu_i(m,p)$$

- We have solved for p^* from free entry condition (supposing $m > 0$). So now want to find measure of entrants m^* that solves

$$Y(m,p^*) = \sum_{i=1}^{n} y_i(p^*)\mu_i(m,p^*) = D(p^*)$$

- **Trick:** $\mu(m,p)$ is linear in m, so write $\mu(m,p) = m \times \mu(1,p)$ and solve for m^* as

$$m^* = \frac{D(p^*)}{Y(1,p^*)} = \frac{D(p^*)}{\sum_{i=1}^{n} y_i(p^*)\mu(1,p^*)}$$

Aside: corner solutions

- If $m^* = 0$ there is *no entry*
- Then, in stationary equilibrium, must also have *no exit*
- Stationary distribution of firms just given by stationary distribution of Markov chain

$$\mu_i = \bar{f}_i$$

- Then market clears if

$$Y(p) = \sum_{i=1}^{n} y_i(p)\bar{f}_i = D(p)$$

Solve for p^* (no longer use free-entry condition to determine p^*)

Numerical example

- Suppose preferences and technology

$$y = zn^{\alpha}, \qquad D(p) = \bar{D}/p$$

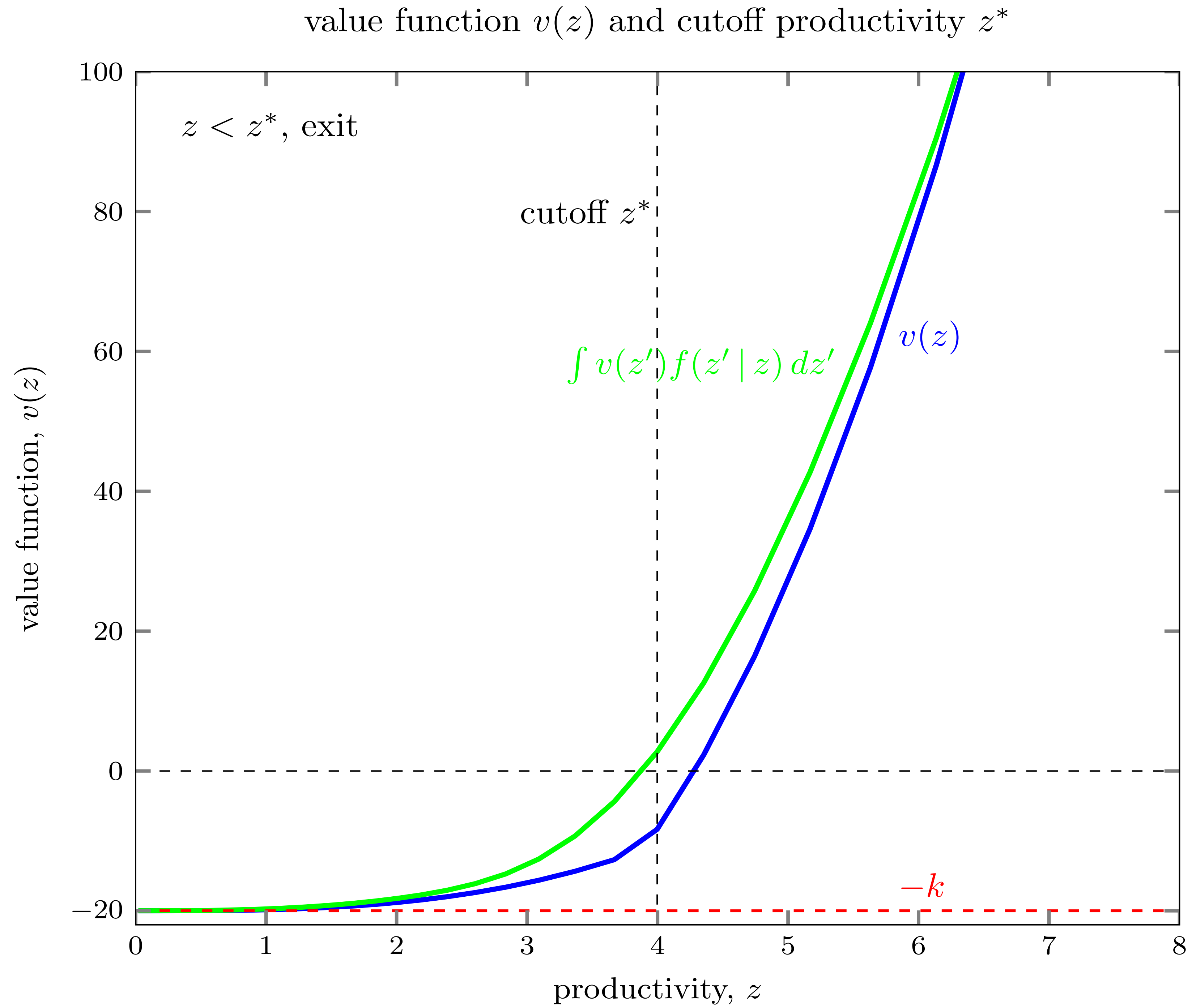
- And that firm productivity follows AR(1) in logs

$$\log z_{t+1} = (1-\rho)\log \bar{z} + \rho \log z_t + \sigma \varepsilon_{t+1}$$

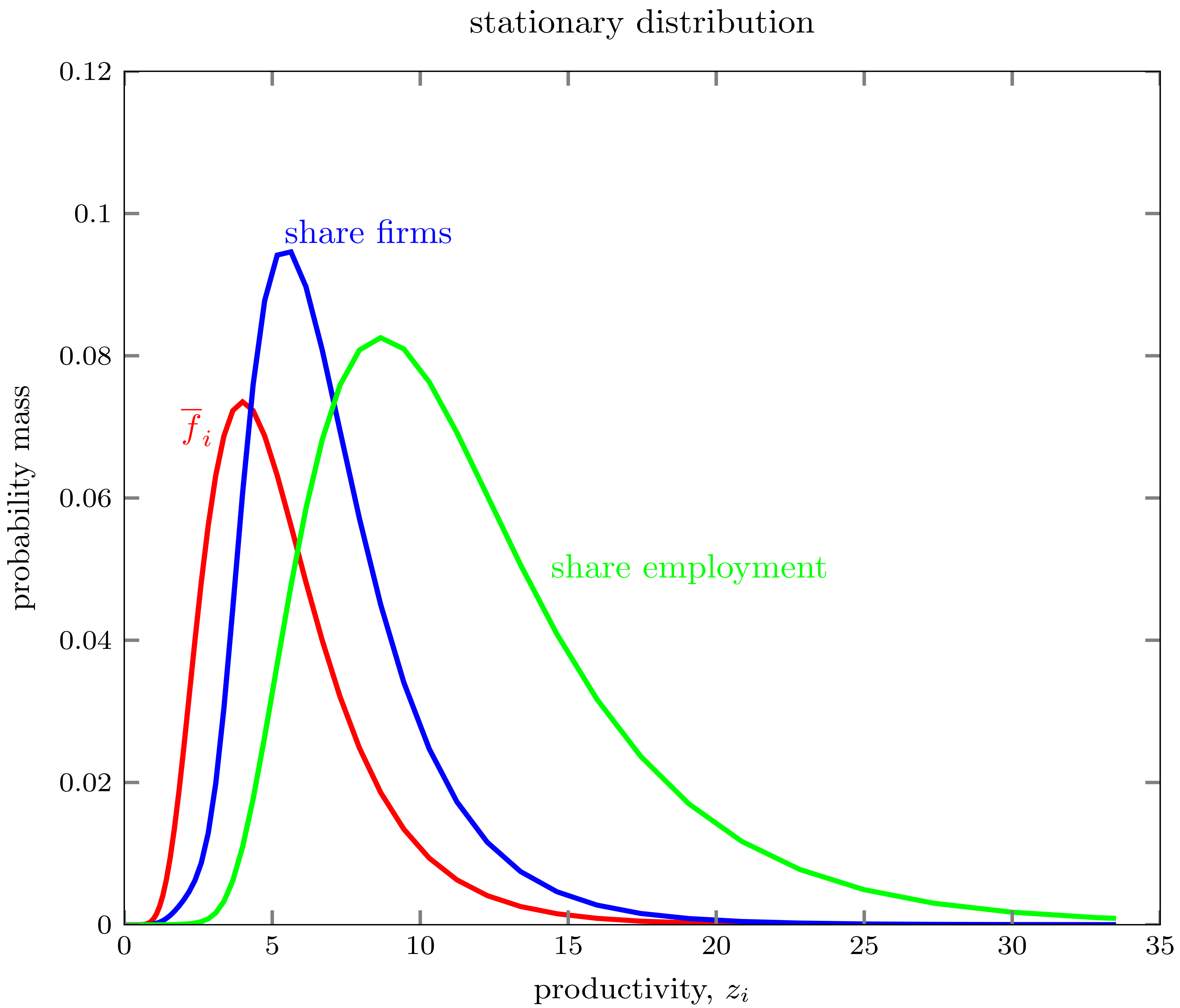
- Parameter values (period length 5 years)

$$\begin{aligned} \alpha = 2/3, \quad \beta = 0.80, \quad k = 20, \quad k_e = 40 \\ \log \bar{z} = 1.40, \quad \sigma = 0.20, \quad \rho = 0.9, \quad \bar{D} = 100 \end{aligned}$$

- Approximate AR(1) with Markov chain on 101 nodes



value function $v(z)$ and cutoff productivity z^*
$z < z^*$, exit
cutoff z^*
$\int v(z')f(z' \mid z)\,dz'$
$v(z)$
$-k$
value function, $v(z)$
productivity, z
100
80
60
40
20
0
−20
0
1
2
3
4
5
6
7
8



stationary distribution
share firms
$\overline{f}_i$
share employment
probability mass
productivity, z_i
0
0.02
0.04
0.06
0.08
0.1
0.12
0
5
10
15
20
25
30
35

Next class

- Hopenhayn and Rogerson (1993)
- Nonconvex labor adjustment costs and job turnover
- Firm-level employment is an endogenous state variable