Macroeconomics

Lecture 18: incomplete markets, part four

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This class

- Krusell-Smith (1998)
  - heterogeneous agent production economy
  - aggregate risk
  - time-varying wealth distribution, high-dimensional state variable
  - approximate aggregation

Firms

- Representative firm with production function

$$Y_t = z_t F(K_t, N_t) = z_t K_t^{\alpha} N_t^{1-\alpha}, \qquad 0 < \alpha < 1$$

- Aggregate productivity z_t follows a 2-state Markov chain with

$$\pi(z' \,|\, z) = \text{Prob}[z_{t+1} = z' \,|\, z_t = z]$$

- Unlike Aiyagari model, factor prices will not be constant

$$r_t = \alpha z_t \left(\frac{K_t}{N_t}\right)^{\alpha-1}$$

and

$$w_t = (1-\alpha) z_t \left(\frac{K_t}{N_t}\right)^{\alpha}$$

Households

- Continuum $i \in [0,1]$ of heterogeneous households, maximize

$$\mathbb{E}\left\{\sum_{t=0}^{\infty} \beta^t u(c_{it})\right\}, \qquad 0 < \beta < 1$$

subject to the budget constraints

$$c_{it} + k_{it+1} \leq r_t k_{it} + w_t n_{it} + (1-\delta) k_{it}$$

where r_t denotes the rental rate of capital

- Also face the constraint

$$k_{it+1} \geq 0$$

(no short-selling capital)

Idiosyncratic risk

- Idiosyncratic labor endowment risk

$$n_{it} \in \{n_l, n_h\}$$

with $n_{it} = n_l$ (unemployed) or $n_{it} = n_h$ (employed)

- *Law of large numbers*: let $\pi(n \,|\, z)$ denote the population fraction of individuals with n if aggregate state is z

- Idiosyncratic risk is *correlated* with the aggregate state. Probability of being unemployed is greater when aggregate productivity is low

Idiosyncratic risk

- Joint transition probability

$$\pi(n', z' \,|\, n, z) = \text{Prob}[n_{t+1} = n', z_{t+1} = z' \,|\, n_t = n, z_t = z]$$

(4 × 4 since each of $n' \,|\, n$ and $z' \,|\, z$ is 2 × 2)

- Consistency requires the adding-up conditions

$$\pi(z' \,|\, z) = \sum_{n'} \pi(n', z' \,|\, n, z), \qquad \text{for all } n$$

and

$$\pi(n' \,|\, z') = \sum_{n} \frac{\pi(n', z' \,|\, n, z)}{\pi(z' \,|\, z)} \pi(n \,|\, z)$$

State variables

- Individual state variables

$$k_{it}\ ,\ n_{it}$$

- Aggregate state variables

$$z_t\ ,\ \mu_t(k_{it}, n_{it})$$

where $\mu_t(k_{it}, n_{it})$ denotes the time-t joint distribution of capital and labor across households

- Unlike the Aiyagari model, distribution evolves over time. For now, write this as

$$\mu_{t+1} = H_t(\mu_t, z_{t+1})$$

(depends on z_{t+1} since fraction of individuals with $n_{t+1} = n$ depends on z_{t+1})

Dynamic programming problem

- Bellman equation for an agent of type k, n given z, μ

$$v(k, n\,;\, z, \mu) = \max_{k' \geq 0} \left[u(c) + \beta \sum_{n'} \sum_{z'} v(k', n'\,;\, z', \mu')\, \pi(n', z' \,|\, n, z) \right]$$

subject to

$$c + k' \leq r(z, \mu)k + w(z, \mu)n + (1 - \delta)k$$

and the law of motion for the distribution μ

$$\mu' = H(\mu, z, z')$$

- Let $k' = g(k, n\,;\, z, \mu)$ denote the policy function implied by the maximization on the RHS of the Bellman equation

- Notice that z, μ matter for individual problem only through factor pricing functions $r(z, \mu)$ and $w(z, \mu)$

Recursive competitive equilibrium

- A *recursive competitive equilibrium* is a value function $v(k, n; z, \mu)$, policy function $g(k, n; z, \mu)$, pricing functions $r(z, \mu)$, $w(z, \mu)$, and law of motion for the distribution $H(\mu, z, z')$ such that

  (i) taking $r(z, \mu), w(z, \mu)$ as given, $v(k, n; z, \mu)$ and $g(k, n; z, \mu)$ solve the dynamic programming problem for an agent of type k, n

  (ii) $r(z, \mu), w(z, \mu)$ solve the firm's profit maximization problem

$$r(z, \mu) = \alpha z \left(\frac{K}{N}\right)^{\alpha-1}$$

$$w(z, \mu) = (1-\alpha) z \left(\frac{K}{N}\right)^{\alpha}$$

Recursive competitive equilibrium

(iii) markets clear

$$K = \sum_k \sum_n k\mu(k, n)$$

$$N = \sum_k \sum_n n\mu(k, n)$$

(iv) the law of motion $H(\mu, z, z')$ is generated by the policy function $k' = g(k, n; z, \mu)$ and the exogenous Markov chain $\pi(n', z' \,|\, n, z)$

Why do we need to keep track of the distribution?

- Household savings decisions depend on return on capital
- Return on capital depends on rental rate next period

$$r(z', \mu') = \alpha z' \left(\frac{K'}{N'}\right)^{\alpha-1}$$

where

$$K' = \sum_k \sum_n k' \mu'(k, n) = \sum_k \sum_n g(k, n, z, \mu) \mu'(k, n)$$

- In principle, whole distribution μ' matters for K'
- Note N' is essentially exogenous, determined by z' alone

Computational strategy

- Approximate *wealth distribution* with finite vector of moments $\boldsymbol{m}$ (mean, variance, skewness, kurtosis, etc)

- The wealth distribution is the distribution μ after n has been marginalized out

- Write the law of motion for this vector of moments

$$\boldsymbol{m}' = \hat{H}(\boldsymbol{m}, z, z')$$

- In practice, Krusell-Smith focus on the case where only the first moment is used

$$\log K' = a_z + b_z \log K$$

(aggregate K is the first moment of the wealth distribution)

Approximate problem

- Bellman equation for an agent of type k, n given z, K

$$v(k, n\,;\, z, K) = \max_{k' \geq 0} \left[u(c) + \beta \sum_{n'} \sum_{z'} v(k', n'\,;\, z', K')\, \pi(n', z' \,|\, n, z) \right]$$

subject to

$$c + k' \leq r(z, K)k + w(z, K)n + (1 - \delta)k$$

and the law of motion

$$\log K' = a_z + b_z \log K$$

- Dependence on μ has been replaced by single number K

- Coefficients a_z, b_z for each z, to be determined

Solution algorithm

- Start with initial guess at coefficients for law of motion a_z^0, b_z^0
- Solve individual's problem for $v^0(k, n\,;\, z, K)$ and $g^0(k, n\,;\, z, K)$
- Use exogenous Markov chain, law of motion, and individual decision rules to simulate a panel of capital stocks k_{it} and let

$$K_t = \frac{1}{I}\sum_{i=1}^{I} k_{it}$$

- For each z, run the regression

$$\log K_{t+1} = a_z^1 + b_z^1 \log K_t + \varepsilon_{t+1}$$

on the simulated aggregate capital stock data

Solution algorithm

- Check if $\left\|(a_z^0, b_z^0) - (a_z^1, b_z^1)\right\|$ is less than some pre-specified tolerance. If so, stop. If not, update to a new guess and try again

- If the coefficients in the law of motion have converged, then the perceived law of motion is approximately consistent with the actual law of motion generated by aggregating individual decisions

Refinement

- Calculate $\left\|(a_z^0, b_z^0) - (a_z^1, b_z^1)\right\|$ and calculate R^2 in the regression

  $$\log K_{t+1} = a_z^1 + b_z^1 \log K_t + \varepsilon_{t+1}$$

  If $(a_z^0, b_z^0) \approx (a_z^1, b_z^1)$ and R^2 in regression is high, stop. Otherwise, update to a new guess and try again

- If coefficients $(a_z^0, b_z^0) \approx (a_z^1, b_z^1)$ but R^2 remains low, *add more moments* to perceived law of motion

- See Den Haan (2010 JEDC) for discussion of computational strategies for this kind of model

Krusell-Smith example

- Four periods per year
- Time discount factor $\beta = 0.96^{1/4} = 0.99$ per period
- Log utility (CRRA $= 1$)
- Capital share $\alpha = 0.36$
- Depreciation rate $\delta = 0.025$ per period
- Aggregate state

$$z \in \{z_l\,,\, z_h\} = \{0.99\,,\, 1.01\}$$

with symmetric transition matrix

$$\pi(z' \,|\, z) = \begin{pmatrix} 0.875 & 0.125 \\ 0.125 & 0.875 \end{pmatrix}$$

(so expected duration of each state is 8 quarters)

Krusell-Smith example

- Idiosyncratic state

$$n \in \{0.25\,,\, 1\}$$

(unemployed have 25% the income of employed)

- Transition probabilities

$$\pi(n', z' \,|\, n, z) = \pi(n' \,|\, z', n, z)\pi(z' \,|\, z)$$

- Have already specified 2×2 transitions $\pi(z' \,|\, z)$

- Need to specify $\pi(n' \,|\, z', n, z)$ for each of $2 \times 2 \times 2 = 8$ combinations

Transition probabilities

- **Expansion:** average unemployment spell 1.5 quarters

$$\pi(\, n' = n_l \,|\, z' = z_h \,,\, n = n_l \,,\, z = z_h \,) = 1/3$$

$$\pi(\, n' = n_h \,|\, z' = z_h \,,\, n = n_l \,,\, z = z_h \,) = 2/3$$

(i.e., $1/(1-(1/3)) = 1.5$ quarters)

- **Recession:** average unemployment spell 2.5 quarters

$$\pi(\, n' = n_l \,|\, z' = z_l \,,\, n = n_l \,,\, z = z_l \,) = 0.6$$

$$\pi(\, n' = n_h \,|\, z' = z_l \,,\, n = n_l \,,\, z = z_l \,) = 0.4$$

(i.e., $1/(1-(0.6)) = 2.5$ quarters)

Transition probabilities

- Switch from $z = z_h$ to $z' = z_l$, higher prob remain unemployed

$$\pi(\, n' = n_l \,|\, z' = z_l \,,\, n = n_l \,,\, z = z_h \,) = 0.75$$

$$\pi(\, n' = n_h \,|\, z' = z_l \,,\, n = n_l \,,\, z = z_h \,) = 0.25$$

- Switch from $z = z_l$ to $z' = z_h$, lower prob remain unemployed

$$\pi(\, n' = n_l \,|\, z' = z_h \,,\, n = n_l \,,\, z = z_l \,) = 0.25$$

$$\pi(\, n' = n_h \,|\, z' = z_h \,,\, n = n_l \,,\, z = z_l \,) = 0.75$$

- Key idea: best time to find job is when economy is expanding (from recession to boom), worst time is when economy is contracting (from boom to recession)

Aggregate consistency

- Fraction of households unemployed conditional on z

$$\pi(n_l \,|\, z_l) = 0.10, \qquad \pi(n_l \,|\, z_h) = 0.04$$

(10% unemployed in recession, 4% unemployed in boom)

- Finally consistency requires

$$\pi(n' \,|\, z') = \sum_n \frac{\pi(n', z' \,|\, n, z)}{\pi(z' \,|\, z)} \pi(n \,|\, z)$$

to pin down all the elements of the 4×4 transition matrix

Quasi-aggregation result

- Law of motion in approximate equilibrium

$$z = z_h : \quad \log K' = 0.095 + 0.0962 \log K, \qquad R^2 = 0.999998$$

$$z = z_l : \quad \log K' = 0.085 + 0.0965 \log K, \qquad R^2 = 0.999998$$

- Standard deviation of the regression residuals

$$z = z_h : \quad \hat{\sigma}_h = 0.0028\%$$

$$z = z_l : \quad \hat{\sigma}_h = 0.0036\%$$

- The first moment alone does an extremely good job

Discussion

- Adding extra moments would in principle improve forecasts
- But improvement in forecasting performance would be minuscule
- With mean alone, forecasts are very accurate. Even 25 years out, errors only on order of 0.1%
- Utility cost of neglecting higher moments is negligible

Aggregate decision rules

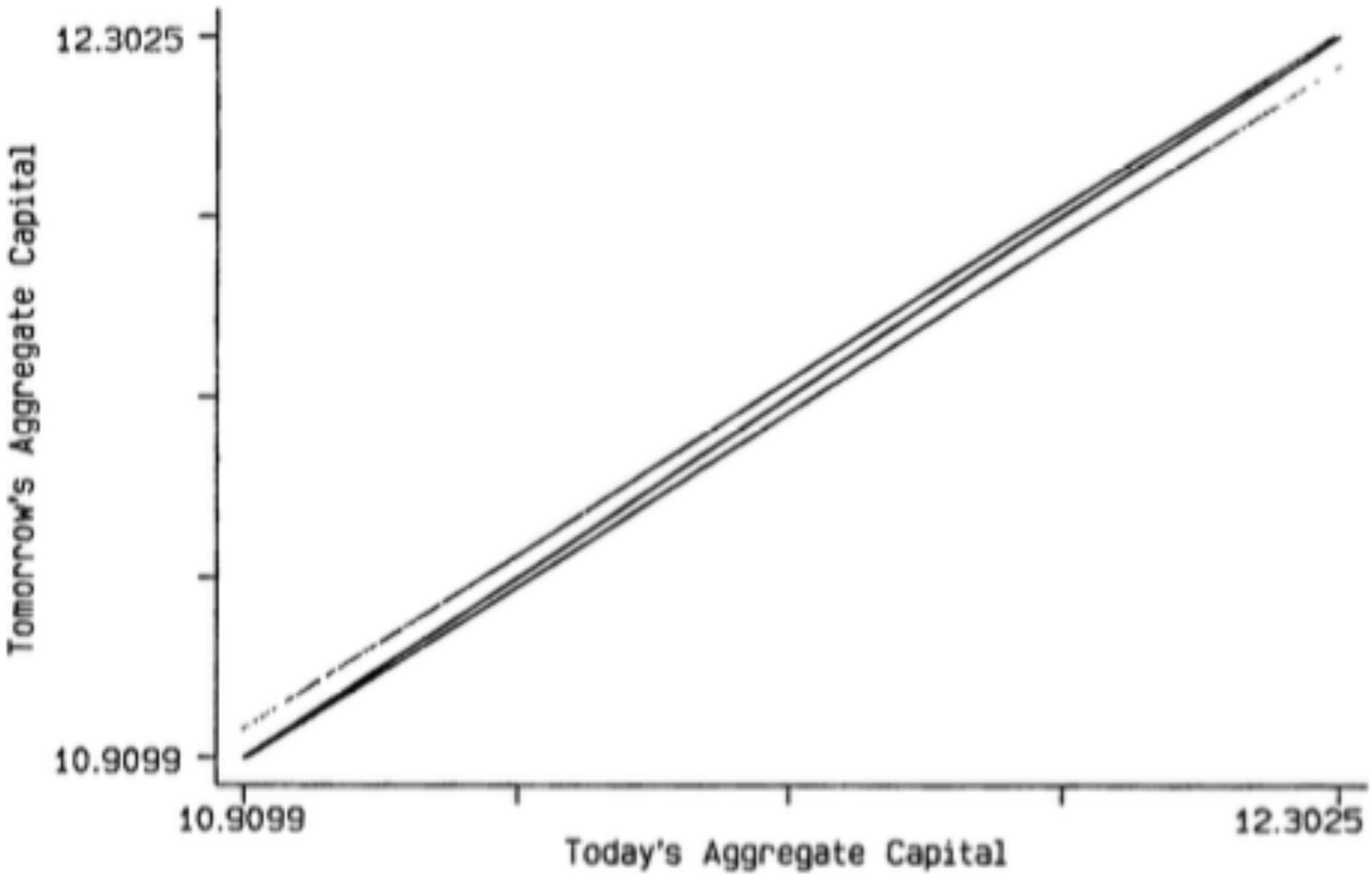
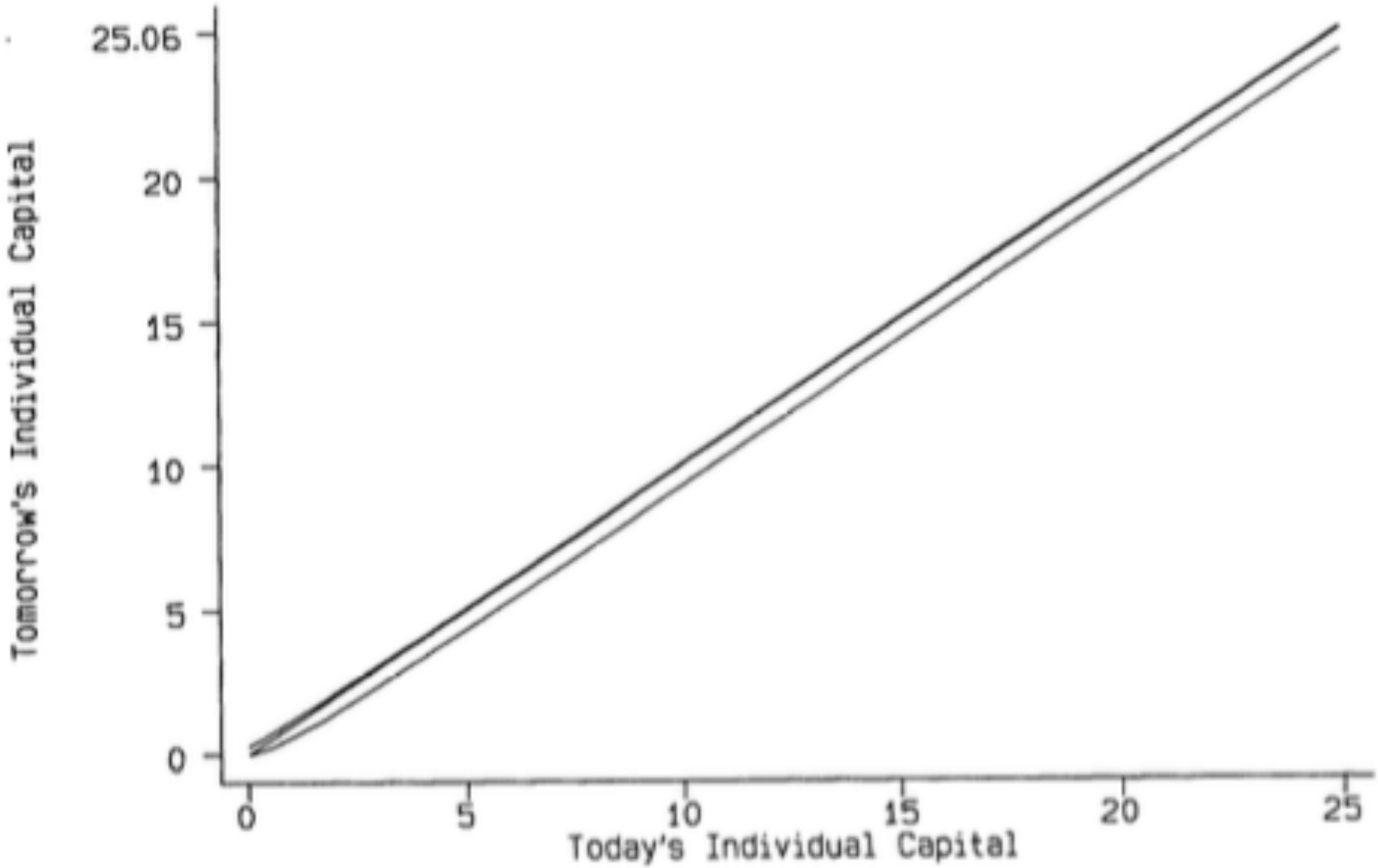


FIG. 1.—Tomorrow's vs. today's aggregate capital (benchmark model)

Individual decision rules



Why is there quasi-aggregation?

- Individual decision rules nearly linear in wealth (capital)
- If they were exactly linear, would get exact aggregation
- There are nonlinearities here, but only for very poor households who don't hold much capital
- Most of the wealth is held by households with a high buffer-stock of savings who all react in the same way to an aggregate shock (same β, same r)

Wealth inequality

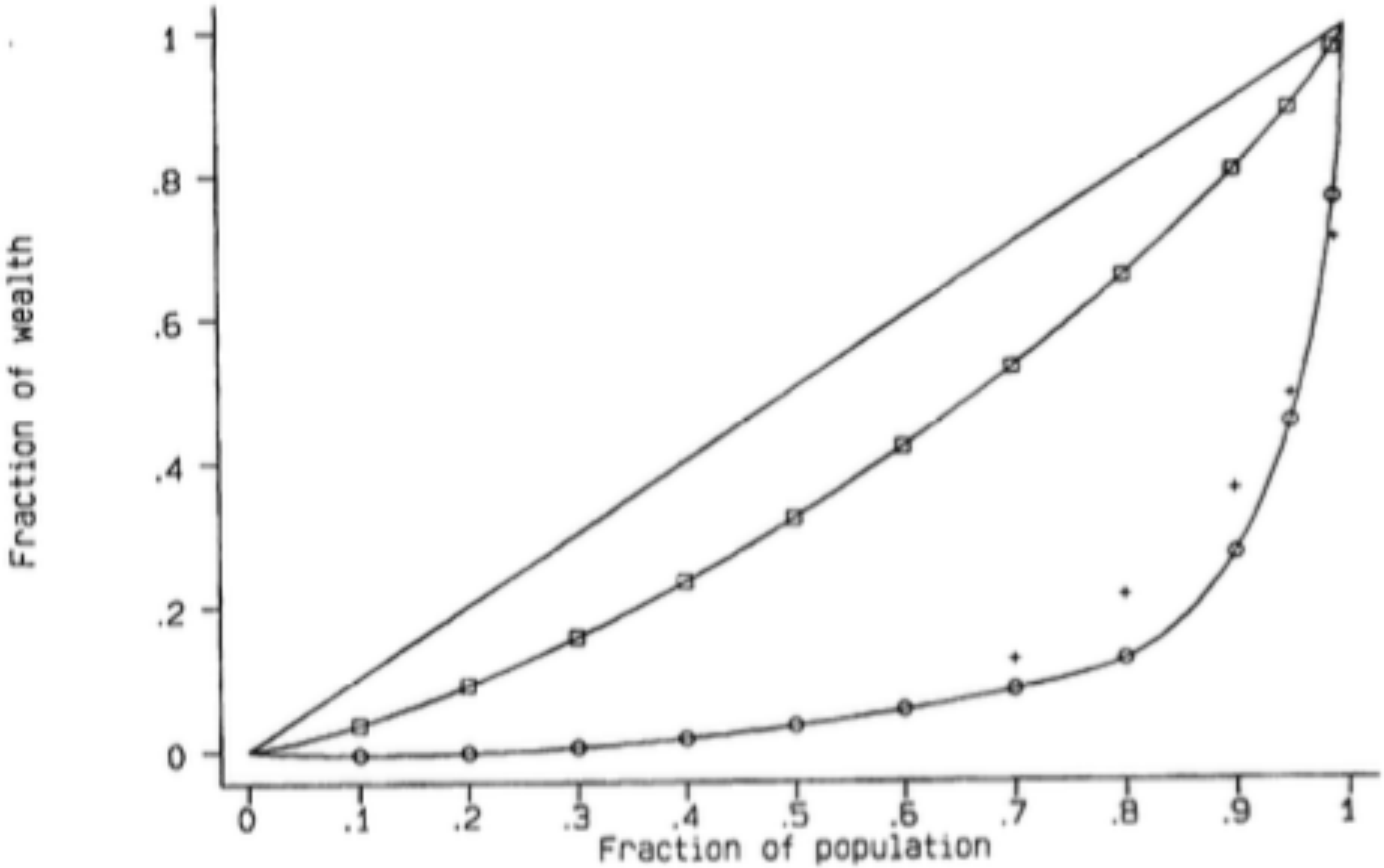


FIG. 3.—Lorenz curves for wealth holdings (+ refers to the data, □ to the benchmark model, and ○ to the stochastic-β model).

Stochastic β model

- Extension with time-varying discount factors

- Three values

$$\beta_1 = 0.9858, \quad \beta_2 = 0.9894, \quad \beta_3 = 0.9930$$

- Long-run distribution putting 80% of population in middle category and 10% in each extreme

- This version of the model leads to much more wealth inequality

Wealth inequality

DISTRIBUTION OF WEALTH: MODELS AND DATA

MODEL	PERCENTAGE OF WEALTH HELD BY TOP 1%	5%	10%	20%	30%	FRACTION WITH WEALTH < 0	GINI COEFFICIENT
Benchmark model	3	11	19	35	46	0	.25
Stochastic-β model	24	55	73	88	92	11	.82
Data	30	51	64	79	88	11	.79

Aggregate fluctuations

AGGREGATE TIME SERIES

Model	Mean(k_t)	Corr(c_t, y_t)	Standard Deviation (i_t)	Corr(y_t, y_{t-4})
Benchmark:				
Complete markets	11.54	.691	.031	.486
Incomplete markets	11.61	.701	.030	.481
$\sigma = 5$:				
Complete markets	11.55	.725	.034	.551
Incomplete markets	12.32	.741	.033	.524
Real business cycle:				
Complete markets	11.56	.639	.027	.342
Incomplete markets	11.58	.669	.027	.339
Stochastic-β:				
Incomplete markets	11.78	.825	.027	.459

Next class

- Hopenhayn (1992)
  - heterogenous firms
  - steady-state model with endogenous entry and exit