Macroeconomics

Lecture 17: incomplete markets, part three

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This class

- Aiyagari (1994)
  - production economy with capital and labor
    (heterogeneous agents version of stochastic growth model)
  - but still no aggregate risk

Firms

- Representative firm with production function

$$Y_t = F(K_t, N_t) = K_t^{\alpha} N_t^{1-\alpha}, \qquad 0 < \alpha < 1$$

- Takes as given rental rate $\tilde{r}_t$ and wage rate w_t

$$\tilde{r}_t = F_{K,t} = \alpha \left(\frac{K_t}{N_t} \right)^{\alpha - 1}$$

$$w_t = F_{N,t} = (1 - \alpha) \left(\frac{K_t}{N_t} \right)^{\alpha}$$

- Aggregate capital evolves according to

$$K_{t+1} = (1 - \delta) K_t + I_t, \qquad 0 < \delta < 1$$

- In equilibrium

$$\tilde{r}_t = r_t + \delta$$

where r_t is the real risk-free return on household savings

Households

- Continuum $i \in [0,1]$ of heterogeneous households, maximize

$$\mathbb{E}\left\{\sum_{t=0}^{\infty}\beta^t u(c_{it})\right\}, \qquad 0 < \beta < 1$$

  subject to the budget constraints

$$c_{it} + a_{it+1} \leq (1+r_t)a_{it} + w_t n_{it}$$

  where r_t is the real risk-free return on their savings

- Also face the borrowing constraint

$$a_{it+1} \geq -\phi$$

  (for some parameter ϕ, to be discussed below)

Labor income risk

- For each individual i, the labor endowment n_{it} follows an exogenous AR(1) process

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$$\log n_{it+1} = (1-\rho)\log \bar{n} + \rho \log n_{it} + \varepsilon_{it+1}, \qquad 0 < \rho < 1$$

  where the innovations ε_{it} are IID over time and across individuals

$$\varepsilon_{it} \sim \text{IID } N(0, \sigma^2)$$

- All the cross-sectional properties of labor income $w_t n_{it}$ come from the exogenous component n_{it}

Markov chain representation

- To simplify the exposition, suppose labor endowment process n_{it} has been approximated by a Markov chain with

$$\pi(n' \,|\, n) = \text{Prob}[n_{it+1} = n' \,|\, n_{it} = n]$$

- Let $\pi(n)$ denote the stationary distribution implied by this chain

- Let $\mu_t(a_{it}, n_{it})$ denote the joint distribution of assets and labor endowments. As in the Huggett model, this is endogenous

Stationary equilibrium

- We again focus on a stationary (steady-state) equilibrium
- In such an equilibrium
  - aggregate variables are constant

    $$w\,,\ r\,,\ N\,,\ K\,,\ \mu(a,n)\,,\ \pi(n)$$

  - but individual-level variables are not constant

    $$c_{it}\,,\ n_{it}\,,\ a_{it}$$

- By contrast with the Huggett model, there is a physical store of value (capital), i.e., aggregate assets are not in zero net supply

Stationary equilibrium

- In such an equilibrium, aggregate labor is simply

$$N = \sum_n n\, \pi(n)$$

which is essentially exogenous

- We also have the basic relationships

$$r + \delta = \alpha \left(\frac{K}{N}\right)^{\alpha - 1} \qquad \Leftrightarrow \qquad K = \left(\frac{\alpha}{r+\delta}\right)^{\frac{1}{1-\alpha}} N \equiv K(r)$$

and

$$w = (1-\alpha)\left(\frac{K}{N}\right)^{\alpha} = (1-\alpha)\left(\frac{\alpha}{r+\delta}\right)^{\frac{\alpha}{1-\alpha}} \equiv w(r)$$

Dynamic programming problem

- Bellman equation for an agent of type a, n given r

$$v(a, n\,;\, r) = \max_{a' \geq -\phi} \left[u(c) + \beta \sum_{n'} v(a', n'\,;\, r)\, \pi(n' \,|\, n) \right]$$

subject to

$$c + a' \leq (1 + r)a + w(r)n$$

- Let $a' = g(a, n\,;\, r)$ denote the policy function implied by the maximization on the RHS of the Bellman equation

- Note that r is constant and that given r individuals do not need to know either K or $\mu(\cdot)$ to solve their problem

- This would not be true if aggregate state was changing (why?)

Stationary equilibrium

- A *stationary equilibrium* is a value function $v(a,n)$, policy function $g(a,n)$, distribution $\mu(a,n)$, price r, and capital stock K such that:

  (i) taking r as given, $v(a,n)$ and $g(a,n)$ solve the dynamic programming problem for an agent of type a,n

  (ii) taking r as given, K solves the firm's profit maximzation problem

  $$K = \left(\frac{\alpha}{r+\delta}\right)^{\frac{1}{1-\alpha}} N$$

  (iii) the asset market clears

  $$K = \sum_a \sum_n g(a,n)\mu(a,n)$$

  (iv) the distribution $\mu(a,n)$ is stationary

  $$\mu(a',n') = \sum_a \sum_n \mathrm{Prob}[a',n' \,|\, a,n]\, \mu(a,n)$$

  where the conditional distribution $\mathrm{Prob}[a',n' \,|\, a,n]$ is given by $a' = g(a,n)$ and $\pi(n' \,|\, n)$

Market clearing

- At the equilibrium r, net asset demand equals supply

$$K = \sum_a \sum_n g(a,n)\mu(a,n)$$

- Assets are in *positive net supply*

  – those with $a' > 0$ are saving at rate r
  – those with $a' < 0$ are borrowing at r

  Unlike the Huggett model, there is a physical store of value and in aggregate there will be more saving than there is borrowing

- If the asset market clears, we also have goods market clearing

$$C \equiv \sum_a \sum_n c(a,n)\mu(a,n), \qquad C + \delta K = K^\alpha N^{1-\alpha}$$

  where $c(a,n)$ denotes the consumption policy implied by $g(a,n)$ and $I = \delta K$ is steady state investment

Solution algorithm

- Start with an initial guess r^0, implies $K(r^0)$ and $w(r^0)$ (given N)
- Solve individual's problem for $v(a, n\,;\, r^0)$ and $g(a, n\,;\, r^0)$ given r^0
- Solve for the stationary distribution $\mu(a, n\,;\, r^0)$ implied by $g(a, n\,;\, r^0)$ and the exogenous $\pi(n' \,|\, n)$
- Compute the error on the market-clearing condition

$$\left\| \sum_a \sum_n g(a, n\,;\, r^0)\mu(a, n\,;\, r^0) - K(r^0) \right\|$$

If this error is less than some pre-specified *tolerance* $\varepsilon > 0$, stop. Otherwise update to r^1 and try again

Updating the return

- We are trying to find r such that the asset market clears

- If for any r^j (for $j = 0, 1, 2, \ldots$) we have

$$\sum_a \sum_n g(a, n\, ;\, r^j)\mu(a, n\, ;\, r^j) > K(r^j)$$

then there is *excess savings* and so we should decrease the return, updating to some $r^{j+1} < r^j$

- Likewise if for any r^j we have

$$\sum_a \sum_n g(a, n\, ;\, r^j)\mu(a, n\, ;\, r^j) < K(r^j)$$

then there is *excess borrowing* and so we should increase the return, updating to some $r^{j+1} < r^j$

Aiyagari's parameterization

- One period per year

- Time discount factor $\beta = 0.96$ per period

- Cobb-Douglas production function with $\alpha = 0.36$

- CRRA $u(c)$ with risk aversion $\{1, 3, 5\}$

- Seven-state Markov chain chosen to replicate AR(1) with innovation standard deviation $\sigma \in \{0.2, 0.4\}$ and persistence $\rho \in \{0, 0.3, 0.6, 0.9\}$

- Benchmark has borrowing constraint $\phi = 0$

Main results

A. Net return to capital in %/aggregate saving rate in % ($\sigma = 0.2$)			
$\rho \backslash \mu$	1	3	5
0	4.1666/23.67	4.1456/23.71	4.0858/23.83
0.3	4.1365/23.73	4.0432/23.91	3.9054/24.19
0.6	4.0912/23.82	3.8767/24.25	3.5857/24.86
0.9	3.9305/24.14	3.2903/25.51	2.5260/27.36
B. Net return to capital in %/aggregate saving rate in % ($\sigma = 0.4$)			
$\rho \backslash \mu$	1	3	5
0	4.0649/23.87	3.7816/24.44	3.4177/25.22
0.3	3.9554/24.09	3.4188/25.22	2.8032/26.66
0.6	3.7567/24.50	2.7835/26.71	1.8070/29.37
0.9	3.3054/25.47	1.2894/31.00	−0.3456/37.63

Special case with IID risk

- Suppose $\pi(n')$ is independent of n and let

$$\hat{a} \equiv a + \phi$$

(so that $a \geq -\phi$ is equivalent to $\hat{a} \geq 0$)

- Let z denote beginning of period '*cash on hand*'

$$z \equiv w(r)n + (1+r)\hat{a} - r\phi$$

so that budget constraint can be written

$$c + \hat{a}' \leq z$$

- For this IID case, dynamic programming is in terms of z only

Special case with IID risk

- Bellman equation for an agent of type z given r

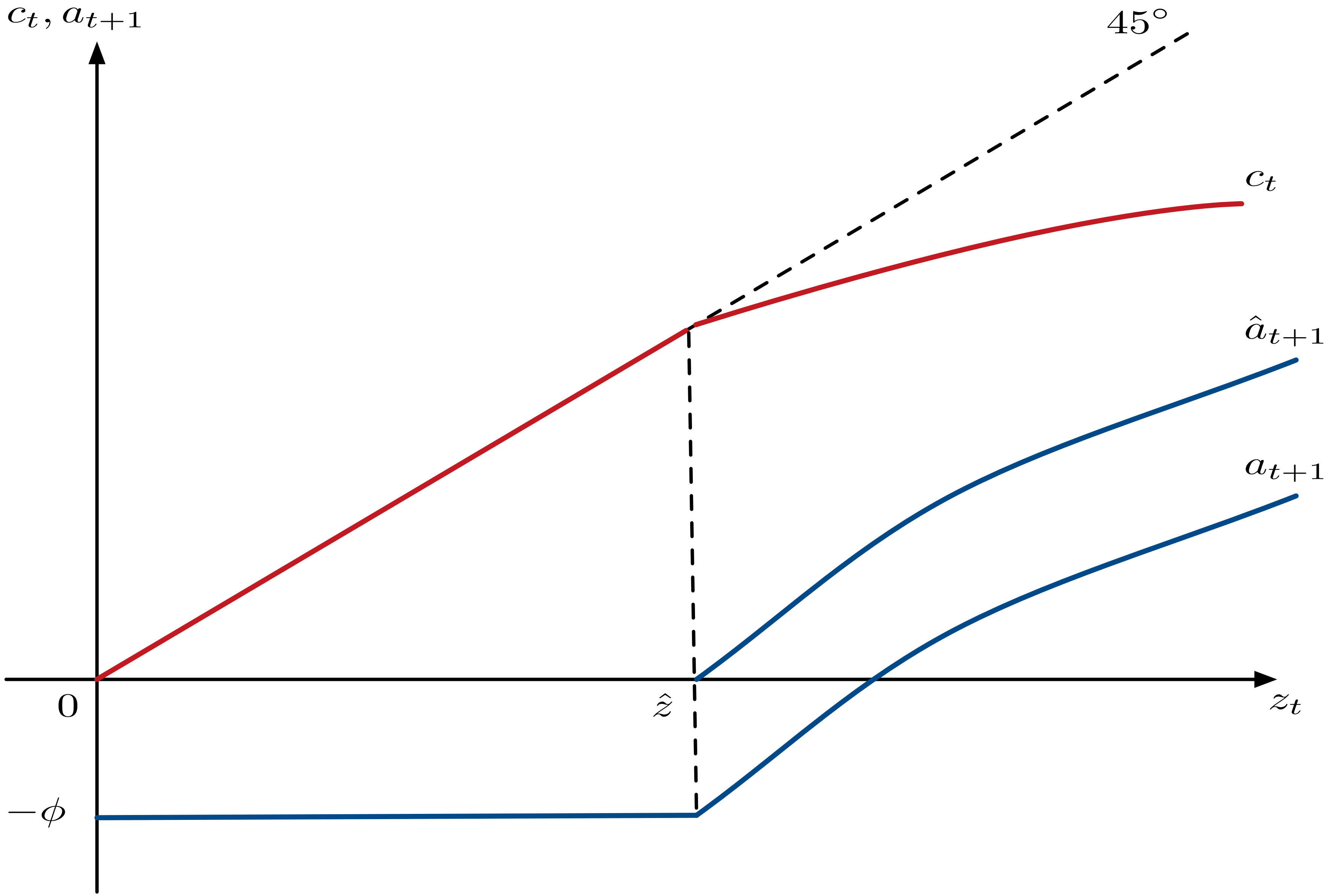
$$v(z\,;\,r) = \max_{\hat{a}' \geq 0} \left[u(c) + \beta \sum_{n'} v(z'\,;\,r)\, \pi(n' \,|\, n) \right]$$

subject to

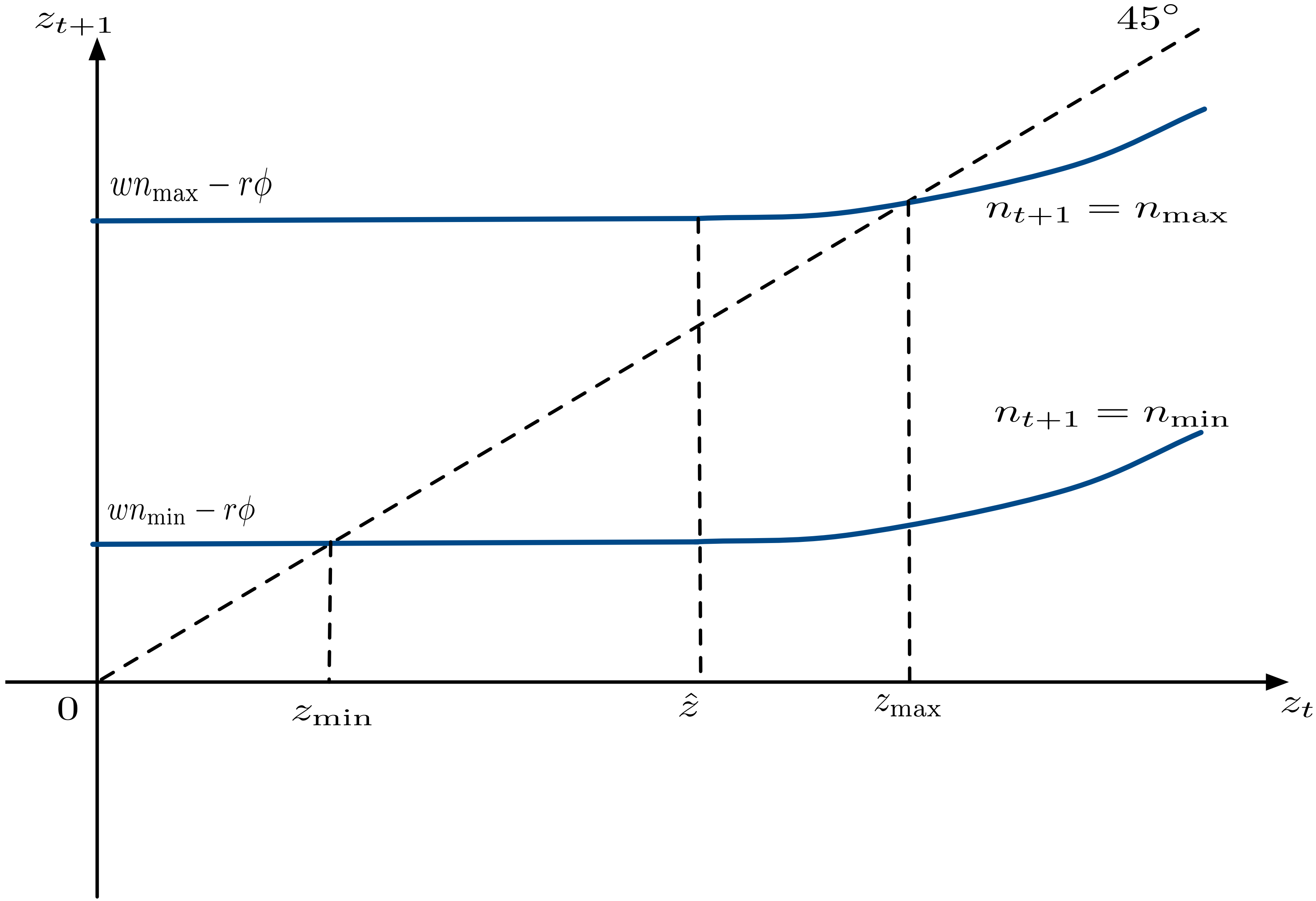
$$c + \hat{a}' \leq z, \qquad z' = w(r)n' + (1+r)\hat{a}' - r\phi$$

- Let $\hat{a}' = g(z\,;\,r)$ denote the policy function implied by the maximization on the RHS of the Bellman equation

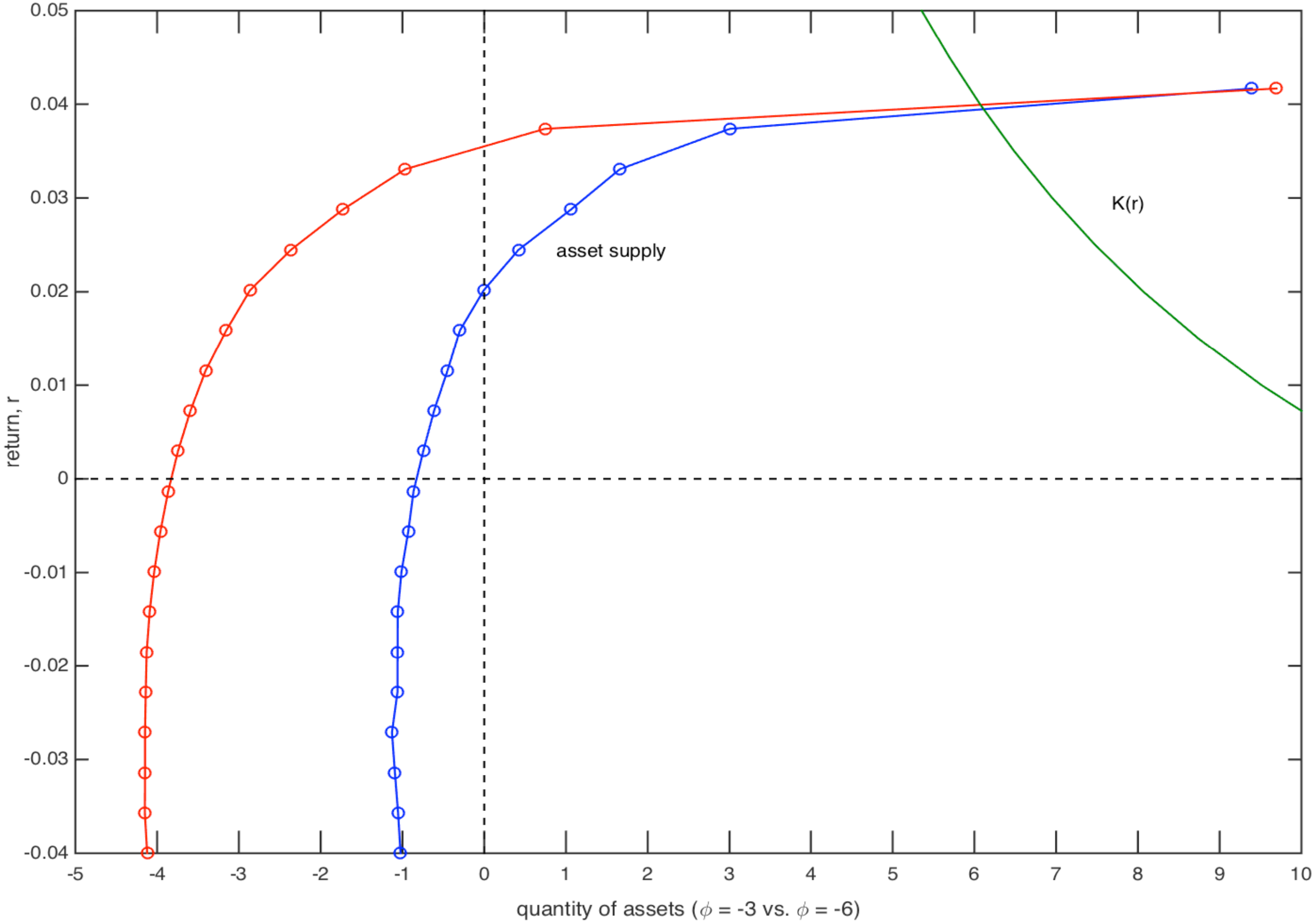
Consumption and assets



Dynamics of total resources z_t



Asset market clearing



Next class

- Krusell-Smith (1998)
  - incomplete markets with aggregate risk
  - i.e., business cycle fluctuations