# Macroeconomics

## Lecture 14: complete markets, part two

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# This class

- Radner (sequential trading) approach to complete markets
- Additional implications when the state is Markov

# Radner (sequence of markets) approach

- At each $t, s^t$ there is trade in a complete set of one-period-ahead contingent claims — also known as *Arrow securities*

- Let $a^i_{t+1}(s^t, s')$ denote the claims individual $i$ purchased at $t, s^t$ for delivery at $t+1, s^{t+1}$ if the state is $s_{t+1} = s'$ is realized

- Let $q_t(s^t, s')$ denote the price of such a claim

- Individual $i$ faces the sequence of one-period budget constraints

$$c^i_t(s^t) + \sum_{s'} q_t(s^t, s') a^i_{t+1}(s^t, s') \leq y^i_t(s^t) + a^i_t(s^{t-1}, s_t)$$

# Assets and liabilities

- If an individual chooses $a^i_{t+1}(s^t, s') > 0$ then they will receive consumption if $s'$ realizes — this is an *asset*

- If an individual chooses $a^i_{t+1}(s^t, s') < 0$ then they must deliver consumption if $s'$ realizes — this is a *liability*, i.e., a form of *debt*

# Radner problem

- Taking prices as given, individuals choose $c_t^i(s^t)$ and the portfolio $a_{t+1}^i(s^t, s')$ to maximize

$$U(c^i) = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u(c_t^i(s^t)) \pi_t(s^t)$$

subject to the sequence of one-period budget constraints

$$c_t^i(s^t) + \sum_{s'} q_t(s^t, s') a_{t+1}^i(s^t, s') \leq y_t^i(s^t) + a_t^i(s^{t-1}, s_t)$$

# Radner problem

- Lagrangian with multipliers $\mu_t^i(s^t) \geq 0$ on the budget constraints

$$\mathcal{L} = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u(c_t^i(s^t))\pi_t(s^t)$$
$$+ \sum_{t=0}^{\infty} \sum_{s^t} \mu_t^i(s^t) \left[ y_t^i(s^t) + a_t^i(s^t) - c_t^i(s^t) - \sum_{s'} q_t(s^t, s') a_{t+1}^i(s^t, s') \right]$$

# Radner problem

- First order condition for $c_t^i(s^t)$ is familiar-looking

$$\beta^t u'(c_t^i(s^t))\pi_t(s^t) = \mu_t^i(s^t)$$

- First order condition for $a_{t+1}^i(s^t, s')$ is

$$q_t(s^t, s')\mu_t^i(s^t) = \mu_{t+1}^i(s^t, s')$$

# Radner problem

- Then eliminating the multipliers on the budget constraints gives

$$q_t(s^t, s_{t+1}) = \beta \, \frac{u'(c^i_{t+1}(s^{t+1}))}{u'(c^i_t(s^t))} \, \pi_{t+1}(s^{t+1} \,|\, s^t)$$

  where $s^{t+1} = (s^t, s_{t+1})$

- Allocations $c^i_t(s^t)$ implied by this sequence-of-markets approach coincide with the Arrow-Debreu allocations

- Recall that in Arrow-Debreu problem

$$q_t^0(s^t) = \beta^t \frac{u'(c_t^i(s^t))}{\mu_i} \pi_t(s^t)$$

where $\mu_i$ denotes multiplier on the lifetime budget constraint

- So the Radner prices can be written

$$q_t(s^t, s_{t+1}) = \frac{q_{t+1}^0(s^t, s_{t+1})}{q_t^0(s^t)}$$

# Equivalence (sketch)

- Can then show that:

  **(i)** given these prices, if $c_t^i(s^t)$ is an Arrow-Debreu allocation then $c_t^i(s^t)$ also satisfies the Radner first order conditions, and that

  **(ii)** such an allocation is budget feasible in the Radner economy, and

  **(iii)** there is no other allocation in the Radner economy that is preferred

- In short, the Radner and Arrow-Debreu allocations coincide

# History-dependence

- Endowments $y_t^i(s^t)$ given by probabilities $\pi_t(s^t)$
- Prices $q_t(s^t, s')$ and portfolios $a_{t+1}^i(s^t, s')$ generally *history-dependent* (even when consumption allocations are not)
- What additional implications follow when shocks are Markov?

# Recursive approach

- Time $t = 0, 1, 2, \ldots$
- Events $s \in S$
- Conditional probabilities $\pi(s' \mid s)$ with interpretation

$$\pi(s' \mid s) = \mathrm{Prob}[s_{t+1} = s' \mid s_t = s]$$

so that

$$\pi_t(s^t) = \pi(s_t \mid s_{t-1})\pi(s_{t-1} \mid s_{t-2}) \cdots \pi(s_1 \mid s_0)\pi(s_0)$$

- Typically set $\pi_0(s_0) = 1$

# Recursive approach

- Individuals $i = 1, 2, \ldots, I$

- Endowments are *time-invariant* functions of current state

$$y^i(s)$$

- Equilibrium consumption allocations are also time-invariant functions of current state only

$$c^i(s)$$

- And equilibrium prices

$$q(s, s') = \beta \, \frac{u'(c^i(s'))}{u'(c^i(s))} \, \pi(s' \,|\, s)$$

# Recursive version of sequential trading

- Individual $i$ has assets $a$ and faces current state $s$
- Choose consumption $c$ and a portfolio of Arrow securities $\boldsymbol{a}(s')$
- Here $\boldsymbol{a}(s')$ denotes a vector specifying a quantity of Arrow securities for each $s' \in S$ (only one of which will pay off)

# Recursive version of sequential trading

- Let $v^i(a, s)$ denote the value function of individual $i$ in state $a, s$

- Satisfies the Bellman equation

$$v^i(a, s) = \max_{c\,,\,\boldsymbol{a}(s')} \left[ u(c) + \beta \sum_{s'} v^i(\boldsymbol{a}(s'), s')\pi(s' \mid s) \right]$$

  where the maximization is subject to the budget constraint

$$c + \sum_{s'} q(s, s')\boldsymbol{a}(s') \leq y^i(s) + a$$

- Let $c^i = g^i(a, s)$ and $\boldsymbol{a}^i(s') = \boldsymbol{h}^i(a, s, s')$ denote the policy functions that achieve the maximum on the RHS of the Bellman equation

# Recursive competitive equilibrium

- A *recursive competitive equilibrium* is a set of prices $Q(s, s')$, a set of value functions $v^i(a, s)$, and a set of policy functions $g^i(a, s)$ and $\boldsymbol{h}^i(a, s, s')$ such that

  **(a)** taking prices as given, the value functions and policy functions solve each individual's problem

  **(b)** the goods market and asset markets clear

# Market clearing

- Let $c_t^i = g^i(a_t^i, s_t)$ and $a_{t+1}^i(s') = h^i(a_t^i, s_t, s')$ iterating forward from the initial asset distribution $a_0^i$

- Goods market clearing requires that in every date and state

$$\sum_i c_t^i = \sum_i y_t^i$$

- Asset market clearing requires that in every date and state

$$\sum_i a_{t+1}^i(s') = 0$$

(for each $s' \in S$)