Macroeconomics

Lecture 13: complete markets, part one

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This class

- Introduction to complete markets general equilibrium
  - time-zero vs. sequential trading arrangements
  - planning allocation vs. equilibrium allocation
  - we do this in a *pure endowment* setting

Setup

- Time $t = 0, 1, 2, \ldots$
- Events $s_t \in S$
- Histories $s^t = (s_0\,,\, s_1\,,\, \ldots\,, s_t) = (s^{t-1}\,,\; s_t)$
- Unconditional probabilities of histories $\pi_t(s^t)$, need not be Markov

Endowments and feasible allocations

- Individuals $i = 1, 2, \ldots, I$
- Individual endowments $y_t^i(s^t)$
- Individual consumption allocations $c_t^i(s^t)$
- *Feasible allocations* satisfy the resource constraint

$$\sum_i c_t^i(s^t) \leq \sum_i y_t^i(s^t), \qquad \text{all } t, s^t$$

- The RHS of this is the *aggregate endowment*

$$Y_t(s^t) \equiv \sum_i y_t^i(s^t)$$

- Let $c^i \equiv \{c_t^i(s^t)\}_{t=0}^{\infty}$ and $y^i \equiv \{y_t^i(s^t)\}_{t=0}^{\infty}$

Preferences

- Individuals rank outcomes using the expected utility criterion

$$U(c^i) \equiv \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \, u(c_t^i(s^t)) \, \pi_t(s^t)$$

- Thus individuals have

  – heterogeneous endowments y^i

  – but *identical utility function* $U(\cdot)$
  (in particular, they agree on $\pi_t(s^t)$ and have same $u(\cdot)$, β)

  – we will relax some of this in coming classes

- As usual, we assume $u'(c) > 0$, $u''(c) < 0$ and $u'(0) = +\infty$

Alternative trading arrangements

(1) Arrow-Debreu (time-zero) approach:

Single enormous market at time $t = 0$, in which there is trade in a *complete set of contingent claims* for all possible histories s^t

At subsequent periods, $t = 1, 2, \ldots,$ agreed-upon trades are carried out but *no further trading occurs*

Alternative trading arrangements

(2) RADNER (sequence of markets) approach:

At each time $t = 0, 1, 2, \ldots$ and history s^t there is a market in which there is trade in a *complete set of contingent claims* for all possible nodes $s^{t+1} = (s^t, s_{t+1})$ that immediately follow s^t

In other words, there is the possibility of *dynamic trading*, contingent on the realized history s^t

Discussion

- Roughly speaking, the Arrow-Debreu time-zero approach has many more assets but many fewer trading dates than Radner sequence-of-markets approach

- Perhaps confusingly, the one-period-ahead contingent claims in the sequence-of-markets approach are known as *Arrow securities*

- It turns out that these two approaches yield identical allocations

Pareto problem

- Consider the problem of a benevolent social planner
- Planner chooses c^i for $i = 1, ..., I$ to maximize the welfare criterion

$$W = \sum_i \lambda_i U(c^i)$$

where $\lambda_i \geq 0$ for $i = 1, ..., I$ are a set of nonnegative *Pareto weights*

- Planner takes as given the resource constraints

$$\sum_i c_t^i(s^t) \leq \sum_i y_t^i(s^t), \qquad \text{for all } t, s^t$$

- A solution to this problem is *Pareto efficient* — i.e., no individual can be made better off without another being made worse off
- By varying the vector of $\boldsymbol{\lambda}$ we can trace out the set of Pareto efficient allocations

Pareto problem

- Lagrangian with stochastic multiplier $\theta_t(s^t) \geq 0$ for each constraint

$$\mathcal{L} = \sum_i \lambda_i \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u(c_t^i(s^t))\pi_t(s^t) + \sum_{t=0}^{\infty} \sum_{s^t} \theta_t(s^t) \sum_i \left[y_t^i(s^t) - c_t^i(s^t)\right]$$

- This can be written more compactly as

$$\mathcal{L} = \sum_i \sum_{t=0}^{\infty} \sum_{s^t} \left\{\lambda_i \beta^t u(c_t^i(s^t))\pi_t(s^t) + \theta_t(s^t)\left[y_t^i(s^t) - c_t^i(s^t)\right]\right\}$$

which reveals that, from the planner's point of view, this is really a sequence of static problems (why?)

Pareto problem

- First order conditions for $c_t^i(s^t)$ are

$$\lambda_i \, \beta^t \, u'(c_t^i(s^t)) \, \pi_t(s^t) = \theta_t(s^t), \qquad \text{for all } i, t, s^t$$

- Hence taking the ratio of these for individual i and individual 1

$$\frac{\lambda_i}{\lambda_1} \frac{u'(c_t^i(s^t))}{u'(c_t^1(s^t))} = 1$$

- We can invert this to write $c_t^i(s^t)$ in terms of $c_t^1(s^t)$, namely

$$c_t^i(s^t) = u'^{\,-1}\left(\frac{\lambda_1}{\lambda_i} \, u'(c_t^1(s^t))\right)$$

Pareto problem

- We can plug this into the resource constraint to get

$$\sum_i u'^{\,-1}\left(\frac{\lambda_1}{\lambda_i}\, u'(c_t^1(s^t))\right) = Y_t(s^t), \qquad \text{for all } t, s^t$$

- This is a single nonlinear equation in $c_t^1(s^t)$ that we can solve

- Once we have found $c_t^1(s^t)$ can then recover $c_t^i(s^t)$ for all other i

Properties of the solution

- Solutions have the form

$$c_t^i = f(\,\lambda_i\,,\,Y_t\,;\,\boldsymbol{\lambda}\,)$$

- Time-invariant function $f(\cdot)$

- *Distribution-free*, cross-sectional distribution of endowments y_t^i realized at t does not matter, only aggregate Y_t matters

- *History-free*, current Y_t is a sufficient statistic for whole history

- Parameterized by vector $\boldsymbol{\lambda}$ of exogenous Pareto weights and $u'(\cdot)$

Arrow-Debreu (time-zero) approach

- Let $q_t^0(s^t)$ denote the price at date $t = 0$ of a claim to one unit of consumption for delivery at t, s^t (superscript 0 refers to date of trade, subscript t refers to date trade is carried out)

- Taking prices $q_t^0(s^t)$ as given, individuals choose consumption plans $c_t^i(s^t)$ to maximize

$$U(c^i) \equiv \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \, u(c_t^i(s^t)) \, \pi_t(s^t)$$

subject to the single budget constraint

$$\sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) c_t^i(s^t) \leq \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) y_t^i(s^t)$$

- RHS is the time-zero value of their future endowments, i.e., their *initial wealth*. The LHS is the time-zero value of consumption

Equilibrium concept

- A *price system* is a sequence of functions $q = \{q_t^0(s^t)\}_{t=0}^{\infty}$. An *allocation* is a collection of sequences of functions $c^i = \{c_t^i(s^t)\}_{t=0}^{\infty}$

- A *competitive equilibrium* is a price system q and a *feasible* allocation c^i such that, taking q as given, the allocation c^i solves each individual's problem

Arrow-Debreu problem

- Lagrangian with *single multiplier* $\mu_i \geq 0$ on budget constraint

$$\mathcal{L} = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u(c_t^i(s^t))\pi_t(s^t)$$
$$+ \mu_i \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t)\left[y_t^i(s^t) - c_t^i(s^t)\right]$$

- Again, this can be written more compactly as

$$\mathcal{L} = \sum_{t=0}^{\infty} \sum_{s^t} \left\{\beta^t u(c_t^i(s^t))\pi_t(s^t) + \mu_i\, q_t^0(s^t)\left[y_t^i(s^t) - c_t^i(s^t)\right]\right\}$$

Arrow-Debreu problem

- First order conditions for $c_t^i(s^t)$ are

$$\beta^t\, u'(c_t^i(s^t))\,\pi_t(s^t) = \mu_i\, q_t^0(s^t), \qquad \text{for all } t, s^t$$

(implies a demand curve for c_t^i in terms of μ_i, prices $q_t^0(s^t)$ etc)

- To find the equilibrium allocation, begin by taking the ratio of these for individual i and individual 1

$$\frac{u'(c_t^i(s^t))}{u'(c_t^1(s^t))} = \frac{\mu_i}{\mu_1}$$

- We can invert this to write $c_t^i(s^t)$ in terms of $c_t^1(s^t)$, namely

$$c_t^i(s^t) = u'^{\,-1}\left(\frac{\mu_i}{\mu_1}\,u'(c_t^1(s^t))\right)$$

Arrow-Debreu problem

- For this to be an equilibrium allocation it must be feasible

$$\sum_i u'^{-1}\left(\frac{\mu_i}{\mu_1}u'(c_t^1(s^t))\right) = Y_t(s^t)$$

This is a single nonlinear equation in $c_t^1(s^t)$ that we can solve

- Once we have found $c_t^1(s^t)$ can then recover $c_t^i(s^t)$ for all other i. This gives

$$c_t^i = g(\mu_i\,,\,Y_t\,;\boldsymbol{\mu})$$

- Again a time-invariant function, history matters only through realization of aggregate endowment Y_t etc

- But this is *not* a solution to the general equilibrium problem

Arrow-Debreu problem

- Still need to solve for the vector of multipliers $\boldsymbol{\mu}$

- For each individual i, evaluate the budget constraint at $c_t^i = g(\mu_i\,, Y_t\,; \boldsymbol{\mu})$ to get

$$\sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t)\, g(\mu_i, Y_t\,; \boldsymbol{\mu}) = \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) y_t^i(s^t), \qquad \text{all } i$$

- Represents a system of I nonlinear equations in I unknowns μ_i. Since μ_i enters $g(\cdot)$ in form μ_i/μ_1 can multiply all μ_i by a constant without changing anything (i.e., can choose a normalization)

- Multipliers μ_i depend on cross-sectional distribution of wealth (which is endogenous). A *fixed point problem*

Equilibrium computation (sketch)

Versions of the following procedure are often used:

1. Fix μ_1. Guess a value for the remaining μ_i. Use these guesses to compute a tentative $c_t^i = g(\mu_i, Y_t\,;\,\boldsymbol{\mu})$

2. Recover the price system from

$$q_t^0(s^t) = \beta^t \, \frac{u'(g(\mu_i, Y_t\,;\,\boldsymbol{\mu}))}{\mu_i} \, \pi_t(s^t)$$

(can use any i, say $i = 1$)

3. Given these $q_t^0(s^t)$, solve system of budget constraints for new μ_i

4. Iterate on steps 1-3 until the μ_i converge

This is known as the *Negishi algorithm*

Equilibrium and planning allocations

- If it turns out that $\mu_i = 1/\lambda_i$ then the equilibrium allocation coincides with the planning allocation (corresponding to $\boldsymbol{\lambda}$)

- Put differently, there is a set of planning solutions indexed by the configuration of $\boldsymbol{\lambda}$ and the competitive equilibrium *picks out one particular* solution, the one for which the planner has $\lambda_i = 1/\mu_i$

- Since μ_i will typically be inversely related to individual i's wealth, this is akin to saying the competitive equilibrium picks out the solution for which the planner *gives high weight to wealthy individuals and low weight to poor individuals*

- At these weights, the planner's multipliers (i.e., shadow prices) $\theta_t(s^t)$ coincide with the equilibrium prices $q_t^0(s^t)$

Equilibrium and planning allocations

- In a competitive equilibrium, the multipliers μ_i are endogenous and determined by the distribution of endowments $y^i = \{y_t^i(s^t)\}$ both directly and indirectly via the equilibrium prices q

- Different configurations of y^i imply different configurations of μ_i and hence different allocations

- Put differently, if we have some desired outcome $c^i = \{c_t^i(s^t)\}$ in mind then we could try to find the configuration of $y^i \to \mu_i$ that would deliver c^i as an equilibrium outcome

- In other words, we can obtain other equilibrium allocations by an appropriate redistribution of wealth

Welfare theorems

- This connection between the equilibrium and planning allocations reflects the two "*fundamental theorems of welfare economics*"

- That the equilibrium allocation corresponds to the solution of *a* planning problem is a version of the *first welfare theorem*, that competitive equilibrium allocations are Pareto efficient

- That we can obtain other equilibrium allocations by an appropriate redistribution of wealth is a version of the *second welfare theorem*, that under some mild regularity conditions, any Pareto efficient allocation can be supported by a competitive equilibrium