Macroeconomics

Lecture 10: dynamic programming applications, part one

Chris Edmond

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This class

- Dynamic programming applications, part one
- Introduction to consumption-based asset pricing
  - asset prices in an endowment economy
  - contingent claims
  - implications for asset returns

Overview

- Endowment economy
  - single type of durable asset
    - a fruit *tree*
  - asset delivers flow of nondurable consumption goods (dividends)
    - each tree produces a random harvest of *fruit*
- We want to see how this durable asset is priced in equilibrium

Setup

- Time $t = 0, 1, 2, \ldots$
- Representative consumer, endowed with initial asset holdings k_0
- Stochastic flow of consumption goods y_t per unit of the asset with conditional distribution

$$F(y' \mid y) = \text{Prob}[\, y_{t+1} \leq y' \mid y_t = y]$$

- Price p_t of buying the asset at date t, taken as given
- Price p_t is 'ex-dividend' (asset bought at t, first dividend at $t + 1$)

Dynamic programming problem

- Representative consumer takes as given a pricing function $p(y)$

- Bellman equation can be written

$$v(k, y) = \max_{k' \geq 0} \left[u(c) + \beta \int v(k', y') \, dF(y' \,|\, y) \right]$$

subject to

$$c + p(y)k' \leq (p(y) + y)k$$

- The RHS of the budget constraint is the consumer's wealth

$$w \equiv (p(y) + y)k$$

- Let $k' = g(k, y)$ denote the policy function implied by the maximization on the RHS of the Bellman equation

Dynamic programming problem

- Alternative Bellman equation using wealth as the state variable

$$v(w, y) = \max_{c \geq 0} \left[u(c) + \beta \int v(w', y') \, dF(y' \,|\, y) \right]$$

subject to

$$w' = R(y', y) \, (w - c)$$

- The term $R(y', y)$ is the gross return on the asset

$$R(y', y) \equiv \frac{p(y') + y'}{p(y)}$$

(also the return on wealth since here there is only one asset)

Recursive competitive equilibrium

- A *recursive competitive equilibrium* is a value function $v(k, y)$, policy function $g(k, y)$ and pricing function $p(y)$ such that:

  (i) taking $p(y)$ as given, $v(k, y)$ and $g(k, y)$ solve the consumer's dynamic programming problem, and

  (ii) the asset market clears

$$g(k, y) = k_0, \qquad \text{for all } k, y$$

- If the asset market clears, the budget constraint implies

$$c + p(y)\, k_0 = (p(y) + y)\, k_0$$

so that we have the goods market clearing condition

$$c = y\, k_0$$

Recursive competitive equilibrium

- Normalize initial asset holdings to $k_0 = 1$
- Then equilibrium consumption allocation is

$$c = y$$

- What are the asset prices implied by this consumption allocation?

Characterizing asset prices

- The first order condition for the consumer can be written

$$u_1(c)p(y) = \beta \int v_1(k', y')\, dF(y' \,|\, y)$$

where it is understood that

$$c = (p(y) + y)k - p(y)g(k, y)$$

and where $u_1(c)$ and $v_1(k, y)$ denote first derivatives

- Writing

$$v(k, y) = u((p(y)+y)k - p(y)g(k, y)) + \beta \int v(g(k, y), y')\, dF(y' \,|\, y)$$

the envelope condition gives

$$v_1(k, y) = u_1(c)(p(y) + y)$$

Characterizing asset prices

- Combining the first order and envelope conditions gives

$$u_1(c) = \beta \int u_1(c') \, \frac{p(y') + y'}{p(y)} \, dF(y' \,|\, y)$$

where it is understood that c, c' are evaluated at the optimum

- Notice that this is the same as

$$u_1(c) = \beta \int u_1(c') \, R(y', y) \, dF(y' \,|\, y)$$

which in our usual time-series notation is

$$u_1(c_t) = \beta \mathbb{E}_t \left\{ u_1(c_{t+1}) R_{t+1} \right\}, \qquad R_{t+1} = \frac{p_{t+1} + y_{t+1}}{p_t}$$

Equilibrium asset prices

- In equilibrium $c = y$ etc so equilibrium asset prices $p(y)$ solve

$$u_1(y) = \beta \int u_1(y') \frac{p(y') + y'}{p(y)} \, dF(y' \,|\, y)$$

  or

$$p(y) = \beta \int \frac{u_1(y')}{u_1(y)} (p(y') + y') \, dF(y' \,|\, y)$$

- The equilibrium pricing function $p(y)$ is a fixed point of this functional equation

- This functional equation is linear so this basically reduces to solving a linear algebra problem (we'll see some examples)

Iterated Euler equations

- In time-series notation we have

$$p_t = \mathbb{E}_t \left\{ \beta \frac{u_1(y_{t+1})}{u_1(y_t)} (p_{t+1} + y_{t+1}) \right\}$$

- But

$$p_{t+1} = \mathbb{E}_{t+1} \left\{ \beta \frac{u_1(y_{t+2})}{u_1(y_{t+1})} (p_{t+2} + y_{t+2}) \right\}$$

Iterated Euler equations

- Substituting for p_{t+1}

$$p_t = \mathbb{E}_t \left\{ \beta \frac{u_1(y_{t+1})}{u_1(y_t)} \left(\mathbb{E}_{t+1} \left\{ \beta \frac{u_1(y_{t+2})}{u_1(y_{t+1})} (p_{t+2} + y_{t+2}) + y_{t+1} \right) \right\} \right\}$$

$$= \mathbb{E}_t \left\{ \beta \frac{u_1(y_{t+1})}{u_1(y_t)} y_{t+1} + \beta^2 \frac{u_1(y_{t+2})}{u_1(y_t)} (y_{t+2} + p_{t+2}) \right\}$$

where the second line uses the *law of iterated expectations*

- More generally we have, iterating forward T times,

$$p_t = \mathbb{E}_t \left\{ \sum_{j=1}^{T} \beta^j \frac{u_1(y_{t+j})}{u_1(y_t)} y_{t+j} \right\} + \mathbb{E}_t \left\{ \beta^T \frac{u_1(y_{t+T})}{u_1(y_t)} p_{t+T} \right\}$$

So in the limit we have

$$p_t = \mathbb{E}_t \left\{ \sum_{j=1}^{\infty} \beta^j \frac{u_1(y_{t+j})}{u_1(y_t)} y_{t+j} \right\} + \mathbb{E}_t \left\{ \lim_{T \to \infty} \beta^T \frac{u_1(y_{t+T})}{u_1(y_t)} p_{t+T} \right\}$$

Iterated Euler equations

- Think of this as

$$\boxed{p_t = \text{fundamental component} + \text{speculative component}}$$

- In *equilibrium*, the speculative component is zero. To see why, suppose not. For example, suppose

  $$\mathbb{E}_t\Big\{\beta^T u_1(y_{t+T})\, p_{t+T}\Big\} > 0$$

  Then the marginal value of selling the asset exceeds the value of consuming its dividends forever

  $$u_1(y_t)p_t > \mathbb{E}_t\left\{\sum_{j=1}^{\infty} \beta^j u_1(y_{t+j})\, y_{t+j}\right\}$$

  So everyone would want to sell the asset, driving its price down

- Likewise if the speculative component is < 0 then everyone would prefer to buy it, driving its price up

Iterated Euler equations

- Thus in this model equilibrium asset prices are given by the fundamental component

$$p_t = \mathbb{E}_t \left\{ \sum_{j=1}^{\infty} \beta^j \frac{u_1(y_{t+j})}{u_1(y_t)} \, y_{t+j} \right\}$$

(i.e., the expected discounted value of the dividend stream)

- Dividends are discounted from $t + j$ back to t using the *stochastic discount factor*

$$M_{t,t+j} = \beta^j \frac{u_1(y_{t+j})}{u_1(y_t)}$$

Example: log utility

- Suppose $u(c) = \log c$ so that $u_1(c) = 1/c$. Then

$$p_t = \mathbb{E}_t \left\{ \sum_{j=1}^{\infty} \beta^j \frac{1/y_{t+j}}{1/y_t} \, y_{t+j} \right\}$$

$$= \mathbb{E}_t \left\{ \sum_{j=1}^{\infty} \beta^j \, y_t \right\} = \frac{\beta}{1-\beta} \, y_t$$

- So, for log utility, the equilibrium pricing function is

$$p(y) = \frac{\beta}{1-\beta} \, y$$

When y is high, consumers seek to smooth consumption by buying assets and asset prices rise to ensure $k = 1$. When y is low, consumers seek to smooth consumption by selling assets and asset prices fall to again ensure $k = 1$

Example: log utility

- Constant price/dividend ratio

$$\frac{p_t}{y_t} = \frac{\beta}{1-\beta} = \frac{1}{\rho}$$

where $\rho = \frac{1}{\beta} - 1$ is the pure rate of time preference

- Capital gains

$$\frac{p_{t+1} - p_t}{p_t} = \frac{y_{t+1} - y_t}{y_t}$$

- Gross return

$$R_{t+1} = \frac{p_{t+1} + y_{t+1}}{p_t} = \frac{1}{\beta}\frac{y_{t+1}}{y_t} = (1+\rho)\frac{y_{t+1}}{y_t}$$

Example: log utility

- In this example, equilibrium price p_t does not depend on properties of expected future y_{t+j}. Why not?

- Suppose y_{t+j} is expected to be high. This will tend to drive up demand for the asset

- But high y_{t+j} means $u_1(y_{t+j})$ is low. This will tend to drive down demand for the asset

- With log utility these two effects *exactly cancel* (c.f., income and substitution effects)

Example: CRRA with IID dividend growth

- Suppose $u_1(c) = c^{-\sigma}$ and $g_{t+1} \equiv y_{t+1}/y_t$ is IID over time. Equilibrium prices are given by

$$p_t = \mathbb{E}_t \left\{ \sum_{j=1}^{\infty} \beta^j \left(\frac{y_{t+j}}{y_t} \right)^{-\sigma} y_{t+j} \right\}$$

Dividing both sides by y_t, price/dividend ratio given by

$$\frac{p_t}{y_t} = \mathbb{E}_t \left\{ \sum_{j=1}^{\infty} \beta^j \left(\frac{y_{t+j}}{y_t} \right)^{1-\sigma} \right\}$$

- Notice that

$$\frac{y_{t+j}}{y_t} = \frac{y_{t+j}}{y_{t+j-1}} \times \cdots \times \frac{y_{t+1}}{y_t} = \prod_{i=1}^{j} g_{t+i}$$

Example: CRRA with IID dividend growth

- Since dividend growth is IID

$$\mathbb{E}_t \left\{ \prod_{i=1}^{j} g_{t+i}^{1-\sigma} \right\} = \left(\mathbb{E}[g^{1-\sigma}]\right)^j = \delta^j, \qquad \delta \equiv \mathbb{E}[g^{1-\sigma}]$$

- So equilibrium price/dividend ratio is

$$\frac{p_t}{y_t} = \sum_{j=1}^{\infty} (\beta\delta)^j = \frac{\beta\delta}{1-\beta\delta}$$

and equilibrium pricing function is

$$p(y) = \frac{\beta\delta}{1-\beta\delta}\, y$$

- Price/dividend ratio again constant etc, but now coefficient depends on g_{t+1} distribution and risk aversion

Discussion

- Individually, a consumer perceives net return on asset to be r_{t+1}
- But social net return on the asset is 0 (resources spent on assets do not deliver more resources in the future)
- The general equilibrium consequence of every individual trying to save at rate r_{t+1} is a social return of 0

Contingent claims

- An *Arrow security* is an asset that delivers one unit of consumption if and only if a particular state is realized

- Let $q(y', y)$ denote the price in state y of an Arrow security that delivers one unit of consumption iff y' is realized next period

- Suppose the representative consumer can trade in a *complete set* of Arrow securities

- Let $\boldsymbol{a}'$ denote the representative consumer's portfolio of Arrow securities with typical element $a(y')$

Dynamic programming problem

- Pricing functions $q(y', y)$ and $p(y)$ taken as given
- Bellman equation can be written

$$v(\boldsymbol{a}, k, y) = \max_{\boldsymbol{a}', k'} \left[u(c) + \beta \int v(\boldsymbol{a}', k', y') \, dF(y' \,|\, y) \right]$$

subject to

$$c + p(y)k' + \int q(y', y)a(y') \, dy' \leq (p(y) + y)k + a(y)$$

- As before, can also use wealth as state variable

Characterizing asset prices

- First order conditions for each $a(y')$ are

$$u_1(c)q(y',y) = \beta v_1(\boldsymbol{a}',k',y')\,f(y'\,|\,y)$$

where $f(y'|y)$ is the density associated with $F(y'|y)$

- First order condition for k' is

$$u_1(c)p(y) = \beta \int v_2(\boldsymbol{a}',k',y')\,dF(y'\,|\,y)$$

- Envelope conditions

$$v_1(\boldsymbol{a},k,y) = u_1(c)$$

and

$$v_2(\boldsymbol{a},k,y) = u_1(c)(p(y)+y)$$

Equilibrium asset prices

- In equilibrium again have $c = y$

- Equilibrium prices of Arrow securities are therefore

$$q(y', y) = \beta \, \frac{u_1(y')}{u_1(y)} \, f(y' \,|\, y)$$

- Equilibrium price of the durable asset again solves

$$p(y) = \beta \int \frac{u_1(y')}{u_1(y)} \, (p(y') + y') \, dF(y' \,|\, y)$$

Pricing other assets

- Consider an asset i that pays $x_i(y')$ in state y'

- By no-arbitrage, this asset will have price equal to

$$q_i(y) = \int q(y', y)\, x_i(y')\, dy'$$

- And so in equilibrium

$$q_i(y) = \beta \int \frac{u_1(y')}{u_1(y)}\, x_i(y')\, dF(y' \,|\, y)$$

(i.e., of the form $q_i = \mathbb{E}[M x_i]$ where M is the one-period SDF)

- Payoff $x_i(y')$ could be anything, e.g., payoffs of some exotic option

Asset returns

- Let $R^i(y', y')$ denote the gross return on such an asset

$$R^i(y', y) = \frac{x_i(y')}{q_i(y)}$$

- Can then restate these conditions in terms of asset returns

$$1 = \beta \int \frac{u_1(y')}{u_1(y)} R^i(y', y) \, dF(y' \,|\, y)$$

- Or in more standard time-series notation

$$1 = \mathbb{E}_t \left\{ \beta \frac{u_1(y_{t+1})}{u_1(y_t)} R^i_{t+1} \right\}$$

Risk free asset

- Consider a *sure claim* to a unit of consumption at next period

- This has $x(y') = 1$ for all y' and has price

$$q_f(y) = \int q(y', y)\, 1\, dy' = \beta \int \frac{u_1(y')}{u_1(y)}\, dF(y' \mid y)$$

and return $R^f(y)$ independent of y' (in this sense it is *risk-free*)

$$R^f(y) = \frac{1}{q_f(y)}$$

- Hence in time-series notation we can write

$$1 = \mathbb{E}_t \left\{ \beta \frac{u_1(y_{t+1})}{u_1(y_t)} R^f_t \right\}$$

Risk-free return is the reciprocal of the expected SDF

$$R^f_t = 1 \Bigg/ \mathbb{E}_t \left\{ \beta \frac{u_1(y_{t+1})}{u_1(y_t)} \right\}$$

Consumption-based asset pricing

- Return on any asset i

$$1 = \mathbb{E}_t\{M_{t+1}R^i_{t+1}\}, \qquad M_{t+1} = \beta\frac{u_1(c_{t+1})}{u_1(c_t)}$$

  where M_{t+1} is the one period stochastic discount factor (SDF)

- Return on a risk-free asset

$$1 = \mathbb{E}_t\{M_{t+1}R^f_t\}$$

- Expanding the expectation of the product gives

$$1 = \mathbb{E}_t\{M_{t+1}\}\mathbb{E}_t\{R^i_{t+1}\} + \mathrm{Cov}_t\{M_{t+1}\,,\, R^i_{t+1}\}$$

Expected excess returns

- Since $R_t^f = 1/\mathbb{E}_t\{M_{t+1}\}$ we can write this as

  $$1 = \frac{1}{R_t^f}\mathbb{E}_t\{R_{t+1}^i\} + \mathrm{Cov}_t\{M_{t+1}\,,\,R_{t+1}^i\}$$

  or

  $$\mathbb{E}_t\{R_{t+1}^i\} - R_t^f = -R_t^f\,\mathrm{Cov}_t\{M_{t+1}\,,\,R_{t+1}^i\}$$

- All assets have an expected return equal to the risk-free return plus a *risk premium* (which may be positive or negative)

Risk premia

- The risk premia are given by

$$-R_t^f \operatorname{Cov}_t\{M_{t+1}\,,\,R_{t+1}^i\} = -\frac{\operatorname{Cov}_t\{u_1(c_{t+1})\,,\,R_{t+1}^i\}}{\mathbb{E}_t\{u_1(c_{t+1})\}}$$

- In general these risk premia are time-varying via the conditioning information (but we will see examples where they are constant)

- What determines risk premia is *not the variance* of returns, but rather how those returns *covary* with consumption

- Investors do not care about the volatility of their portfolio per se, it depends on how that translates to volatility in consumption

Risk premia

- Asset returns that covary negatively with M_{t+1} deliver high payoffs when marginal utility is low — these are a *bad hedge*, will be in low demand and carry a high risk premium

  (assets that covary positively with c_{t+1} make consumption more volatile)

- Asset returns that covary positively with M_{t+1} deliver high payoffs when marginal utility is high — these are a *good hedge*, will be in high demand and carry a low (or negative) risk premium

  (assets that covary negatively with c_{t+1} make consumption less volatile)

Idiosyncratic risk is not priced

- Only that part of an asset return that is correlated with the aggregate M_{t+1} leads to a risk adjustment (positive or negative)

- The idiosyncratic component in returns offers no better (or worse) hedging opportunities and so is not priced

Consumption CAPM

- This is a version of the *capital asset pricing model* (CAPM) except that covariance with the 'market return' is replaced with covariance with the SDF

- This setting where

$$M_{t+1} = \beta \frac{u_1(c_{t+1})}{u_1(c_t)}$$

  is often referred to as the consumption-CAPM

- Although elegant and intuitive, it is difficult to reconcile this model with data on stock and bond returns (huge literatures on the 'equity premium puzzle', the 'risk-free rate puzzle' etc etc)

Next class

- Job search and matching