PhD Topics in Macro: Problem Set #1

Due Monday August 18th in class

1. **Hopenhayn.** Consider the Hopenhayn model discussed in class. On paying a fixed operating cost $k > 0$, an incumbent firm that hires n workers produces flow output $y = an^{\alpha}$ with $0 < \alpha < 1$ where $a > 0$ is firm-level productivity. The productivity of an incumbent firm evolves according to an AR(1) in logs

$$\log a_{t+1} = (1-\rho)\log \bar{a} + \rho \log a_t + \sigma\varepsilon_{t+1}, \qquad 0 < \rho < 1,\, \sigma > 0$$

where $\varepsilon_{t+1} \sim$ IID $N(0,1)$. Firms discount flow profits according to a constant discount factor $0 < \beta < 1$. There is an unlimited number of potential entrants. On paying a sunk entry cost $k_e > 0$, an entrant receives an initial productivity draw $a_0 > 0$ and then starts operating the next period as an incumbent firm. For simplicity, assume that initial productivity a_0 is drawn from the stationary productivity distribution implied by the AR(1) above.

Individual firms take the price p of their output as given. Industry-wide demand is given by the demand function $D(p) = \bar{D}/p$ for some constant $\bar{D} > 0$. Let labor be the numeraire, so that the wage is $w = 1$. Let $\pi(a)$ and $v(a)$ denote respectively the profit function and value function of a firm with productivity a. Let v^e denote the corresponding expected value of an entering firm. Let $\mu(a)$ denote the (stationary) distribution of firms and let m denote the associated measure of entering firms.

   (a) Derive an expression for the profit function $\pi(a)$.

   (b) Set the parameter values $\alpha = 2/3$, $\beta = 0.8$, $k = 20$, $k_e = 39.7$, $\log \bar{a} = 1.3855$, $\sigma = 0.2026$, $\rho = 0.9$ and $\bar{D} = 100$. Use the Tauchen/Hussey approach to discretize the AR(1) process to a Markov chain on 33 nodes. Solve the model on this grid of productivity levels. Calculate the equilibrium price p^* and measure of entrants m^*. Let a^* denote the cutoff level of productivity below which a firm exits. Calculate the equilibrium a^*. Plot the stationary distribution of firms and the implied distribution of employment across firms. Explain how these compare to the stationary distribution of productivity levels implies by the AR(1).

   (c) Now suppose the demand curve shifts, with $\bar{D}$ increasing to 120. How does this change the equilibrium price and measure of entrants? How does this change the stationary distributions of firms and employment? Give intuition for your results.

2. **Hopenhayn/Rogerson.** Consider the Hopenhayn/Rogerson model discussed in class. At the beginning of a period, an incumbent firm knows its existing workforce n and productivity level a. If the firm employs n' workers this period, it produces output with that workforce but pays an adjustment cost $\tau \max[0, n - n']$ in addition to its wage payments. The firm then begins next period with workforce n'. If the firm exits (shrinks employment to $n' = 0$) it pays τn to shut down but avoids paying the standard operating fixed cost $k > 0$. In all other respects the model is the same as in Question 1 above.

(a) Using the same parameter values as in Question 1 above, solve the Hopenhayn/Rogerson model with adjustment costs $\tau = 0, 0.1, 0.2$ and 0.5. For each τ, calculate the equilibrium price p^*, measure of entrants m^* and exit threshold a^*. How do these compare to their counterparts in Question 1? Likewise, for each τ compare the stationary distributions of firms and employment to their counterpart distributions in Question 1.

Hint: be sure to set up a grid of employment choices that nests the employment outcomes from the basic Hopenhayn model and be sure to make your grid of employment sufficiently fine where it needs to be. I obtained reasonable accuracy with a grid of 280 'carefully chosen' employment levels.

(b) Let $\underline{n}(a)$ and $\overline{n}(a)$ denote the boundaries of the employment *inaction region* such that if $n \in (\underline{n}(a), \overline{n}(a))$ then a firm finds $n' = n$ optimal. Calculate these boundaries of the inaction region for each τ. Explain how the boundaries vary with productivity a and how they vary with τ. Give intuition for your answers.

(c) Calculate the marginal product of labor for each firm type. Explain how the marginal product of labor varies across firm size and productivity levels. How does the distribution of marginal products vary with τ? Give intuition for your answers.

(d) Let Y and N denote aggregate output and aggregate employment *in production*. Let $A = Y/N$ denote aggregate labor productivity (output per worker). Calculate Y, N and A for each level of τ and give intuition for your results.

3. **Hopenhayn meets Jovanovic.** Consider the following variant on the Hopenhayn model, inspired by Jovanovic's "Selection and the evolution of industry" (1982, *Econometrica*). On entering, a firm not only draws an initial productivity $a_0 > 0$, as usual, but also independently draws a *time-invariant* and *unobservable* productivity type θ that governs its long-run average productivity. Going forward, each period an incumbent firm draws both its current a, as usual, but also independently draws an additional productivity shock $\log z = \theta + \zeta$ centred on θ with noise $\zeta \sim$ IID $N(0, \sigma_\zeta^2)$. Conditional on z and a an incumbent with n workers produces output

$$y = azn^\alpha$$

Hence firms with high θ will tend to have systematically higher total productivity than firms with low θ. The timing within a period is as follows: incumbents decide to stay or exit, entrants decide to enter or not. Incumbents that continue pay the operating fixed cost $k > 0$, entrants pay $k_e > 0$. After paying k, incumbents then observe a, but must choose employment n *before* observing z — i.e., firms make employment decisions based on their expectations of the current z which in turn depend on their beliefs about their underlying θ type.

All firms begin with the common *prior* that $\theta \sim N(\bar{\theta}, \sigma_\theta^2)$ for some prior mean $\bar{\theta}$ and variance σ_θ^2. Since a firm that has just entered has not yet obtained any new information about θ, a new entrant's initial choice of n is still only based on its prior. After n has been chosen and y observed, each incumbent can deduce this period's z and then updates its beliefs about θ before the next period begins. In all other respects the model is the same as in Question 1 above.

(a) Suppose an incumbent firm has been operating for s periods. Let $x := \mathbb{E}[\theta \,|\, z^s]$ denote a firm's conditional expectation for θ if it has observed a history of signals $z^s := (z_1, ..., z_s)$. Let $P(x' \,|\, x, s)$ denote the CDF for next period's expectation x' conditional on this period's x and the number of signals s. Use Bayes's Rule to derive a formula for $P(x' \,|\, x, s)$.

Hint: the normal distribution is *self-conjugate*, a normal prior distribution and normal likelihood imply a normal posterior distribution.

(b) Let $\pi(x, a)$ denote the static expected profit function of a firm with current expectation x and productivity a. Derive a formula for $\pi(x, a)$. Let $v(x, a, s)$ denote the associated value function of an incumbent firm that has observed s signals. Write down the Bellman equation that this value function has to satisfy. Let v^e denote the value of entry. How does this relate to $v(x, a, s)$?

(c) Let $\mu(x, a, s)$ denote the stationary distribution of firms across x, a, s and let m denote the measure of entering firms. In a stationary equilibrium the distribution $\mu(x, a, s)$ has to satisfy a fixed point condition. Derive and explain that condition.

(d) Outline a computational procedure for solving this model. In what sense do you expect this model to feature 'noisy selection'?

(e) Explain intuitively the role of the a shocks in this model. How, if at all, will the equilibrium be different if there are no a shocks? [*Hint*: consider a firm's beliefs about θ as $s \to \infty$].