PhD Topics in Macroeconomics

Lecture 5: innovation and firm dynamics, part one

Chris Edmond

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This lecture

- Review of ‘Schumpeterian’ growth theory
  - quality ladders
  - endogenous growth via creative destruction
- Appendix: review of some continuous time tools
  - continuous time Bellman equations
  - Poisson processes

Model overview

- Goods are 'horizontally' and 'vertically' differentiated
- Vertical differentiation via quality differences
  - higher quality varieties deliver more utility per unit consumption
- Quality improvements arrive according to Poisson process
- Arrival rate of quality improvements is endogenous
  - gives rise to aggregate growth

Quality ladder model

- Continuous time $t \geq 0$

- Representative household

$$U = \int_0^\infty e^{-\rho t} \log C_t \, dt, \qquad \rho > 0$$

- Aggregate consumption C_t depends on

  - $j \in [0, 1]$ continuum horizontally differentiated varieties
  - $k \in \{0, 1, \ldots, J_t(j)\}$ discrete vertically differentiated vintages of j

  - *state-of-the-art* vintage $J_t(j)$ for each horizontal variety j

- Let $z(j, k)$ denote *quality* and $x_t(j, k)$ denote *quantity* of variety j, k

Aggregate consumption

- Instantaneous utility

$$\log C_t = \int_0^1 \log \Big[\sum_{k=0}^{J_t(j)} z(j,k) x_t(j,k) \Big] \, dj$$

- Note: *imperfect* horizontal differentiation (elasticity of subs. $= 1$) but *perfect* vertical differentiation (elasticity of subs. $= \infty$)

- Let $q > 1$ denote the size of the *quality step*, i.e., for each j

$$z(j,k) = q \, z(j,k-1) \qquad k = 1, 2, \ldots, J_t^*(j)$$

- Choose physical units for each variety so that $z(j,0) = 1$ for all j. Then simply $z(j,k) = q^k$ for all j

Expenditure

- Let P_t denote aggregate price index associated with C_t and let $E_t = P_t C_t$ denote aggregate expenditure

- Let $k_t^*(j)$ denote variety that charges lowest price per unit quality

- Demand for variety j, k is then

$$x_t(j,k) = \begin{cases} \dfrac{E_t}{p_t(j,k)} & \text{if } k = k_t^*(j) \\ 0 & \text{otherwise} \end{cases}$$

- Aggregate expenditure satisfies the intertemporal Euler equation

$$\frac{\dot{E}_t}{E_t} = r_t - \rho, \qquad \Leftrightarrow \qquad \frac{\dot{C}_t}{C_t} = r_t - \rho - \frac{\dot{P}_t}{P_t}$$

Expenditure

- Let $E_t = 1$ be the numeraire. Then from the Euler equation

$$r_t = \rho \qquad \Leftrightarrow \qquad \frac{\dot{C}_t}{C_t} = -\frac{\dot{P}_t}{P_t}$$

- And expenditure on variety j, k is

$$p_t(j,k)x_t(j,k) = \begin{cases} 1 & \text{if } k = k_t^*(j) \\ \\ 0 & \text{otherwise} \end{cases}$$

Production

- Wage rate w_t per unit labor engaged in production

- Flow profits from production of j, k

$$\pi_t(j,k) = (p_t(j,k) - w_t)x_t(j,k)$$

(i.e., it takes one unit of labor to produce one unit of output, $x = l$)

- Inelastic aggregate labor supply L. Labor may be employed in *goods production* L_X or in *research* L_R

$$L_X + L_R = L$$

Pricing

- Consider leader firm with state-of-the-art quality and its closest follower, one step behind

- Leader has quality advantage $q > 1$ over follower

- Leader charges *limit price* $p_t(j,k) = qw_t$ to prevent entry

- In symmetric equilibrium only the state-of-the-art quality is sold $k_t^*(j) = J_t(j)$, and all leaders have flow profits

$$\pi_t(j, J_t(j)) = (p_t(j, J_t(j)) - w_t)x_t(j, J_t(j)) = \frac{q-1}{q} =: \pi$$

Innovation

- Any firm can target any product line in an attempt to improve state-of-the-art
- Technology for innovating: cost $c\lambda > 0$ units of labor time delivers flow probability $\lambda\, dt$ of successfully innovating to next step on quality lader, $1 - \lambda\, dt$ flow probability of failure
- Quality steps a Poisson process with intensity $\lambda > 0$
- Memoryless process: no advantages of incumbency, potential entrants can immediately build on the state-of-the-art, etc

Innovation decisions

- Let v_t denote the value of an incumbent firm (to be determined)
- ‘Lottery’ delivers v_t with flow probability $\lambda\,dt$ or 0 with flow probability $(1-\lambda\,dt)$, costs $w_t c\lambda\,dt$ (in units of labor)
- Expected gain from innovation

$$v_t\lambda\,dt + 0(1-\lambda\,dt) - w_t c\lambda\,dt$$

- Gives free-entry condition into innovation

$$v_t \leq w_t c, \qquad \text{with equality whenever } \lambda > 0$$

Incumbents don't invest in innovation

- Will an incumbent try to get *two* steps ahead? If successful, charge price $= q^2 w$ and have sales $= 1/q^2 w$

- Yields flow profits $\pi^2 := 1 - (1/q^2)$. But leader already has flow profits $\pi^1 := 1 - 1/q$ even if no investment in innovation

- Incremental profit for investing incumbent

$$\pi^2 - \pi^1 = \frac{1}{q}\left(\frac{q-1}{q}\right)$$

- Incremental profit for investing entrant

$$\pi^1 - \pi^0 = \left(\frac{q-1}{q}\right) > \pi^2 - \pi^1$$

- With free entry, equilibrium 'cost of capital' will be too high for incumbent firms to find it optimal to invest

Bellman equation for incumbents

- Flow profits π
- Lose incumbency to successful innovator with flow probability $\lambda\, dt$
- Value v_t satisfies continuous time Bellman equation

$$\rho v_t = \pi - \lambda v_t + \dot{v}_t$$

(no aggregate risk, all idiosyncratic risk perfectly diversified)

Equilibrium

- We will focus on a *stationary equilibrium*, constants

$$(v^*\,,\,\lambda^*\,,\,w^*)$$

- Value for incumbents, from steady-state of Bellman equation

$$v = \frac{\pi}{\rho + \lambda}, \qquad \pi = \frac{q-1}{q}$$

for λ to be determined

- Labor market clearing

$$L_X + L_R = L$$

with total labor employed in each sector

$$L_X = \frac{1}{qw}, \qquad \text{and} \qquad L_R = \lambda c$$

Free entry condition $v \leq wc$

- Case 1: $\lambda > 0$ (innovation). Then $v = wc$ and we can write the labor market clearing condition

  $$\frac{(\rho+\lambda)c}{q\pi} + \lambda c = L$$

  or

  $$\lambda^* = \pi\frac{L}{c} - \frac{\rho}{q}$$

  from which we can then recover $v^* = \pi/(\rho+\lambda^*)$ and $w^* = v^*/c$

- Case 2: $\lambda = 0$ (no innovation). Then $L_R = 0$, $L_X = L$ and so $w^* = 1/qL$, $v^* = \pi/\rho$

- Steady-state with innovation exists if quality increments

  $$q > 1 + \rho\frac{c}{L}$$

  (i.e., if large population, low discount rate, or low cost etc)

Aggregate consumption

- Recall aggregate consumption index

$$\log C_t = \int_0^1 \log \Big[\sum_{k=0}^{J_t(j)} q^k \, x_t(j,k) \Big] \, dj$$

using $z(j,k) = q^k$ for all j

- In equilibrium only latest vintage sold, i.e., $x_t(j,k) = 1/qw_t$ for $k = J_t(j)$ and zero otherwise. Hence

$$\log C_t = (\log q) \int_0^1 J_t(j) \, dj - \log(q \, w_t)$$

Aggregate growth

- Use LLN to calculate cross-sectional average

$$\int_0^1 J_t(j)\, dj = \mathbb{E}[J_t]$$

where J_t is a Poisson process with intensity λ^*, so

$$\mathbb{E}[J_t] = \lambda^* t$$

- Hence in a steady-state with $w_t = w^*$, aggregate growth is

$$g^* := \frac{\dot{C}_t}{C_t} = (\log q)\lambda^*$$

Real wages etc

- In steady state, wage w^* is a constant
- But *real wage* w^*/P_t is growing. Recall that

  $$1 = E_t = P_t C_t$$

  so

  $$-\frac{\dot{P}_t}{P_t} = \frac{\dot{C}_t}{C_t} = g^*$$

  Hence real wage is growing at g^* too

- All growth is due to quality upgrading

Aggregate growth

- If interior equilibrium

$$g^* = (\log q)\lambda^* \qquad \text{where} \qquad \lambda^* = \pi\frac{L}{c} - \frac{\rho}{q}$$

- So in this case g^* is

  – increasing in q (directly, and indirectly via λ^*)
  – increasing in L (scale effect)
  – increasing in π (monopoly profits from successful innovation)

  – decreasing in c (barrier to entry/cost of innovation)
  – decreasing in ρ (greater impatience)

- Otherwise, namely if $q < 1 + \rho c/L$, then corner equilibrium with $\lambda^* = 0$ and hence $g^* = 0$ etc

Next

- Innovation and firm dynamics, part two
- Embedding this in a model of firm dynamics
  - Klette and Kortum (2004): Innovating firms and aggregate innovation, *Journal of Political Economy*.
- Integrated treatment of firm heterogeneity and growth via creative destruction

Appendix to Lecture 5

Review of continuous time Bellman equations, Poisson processes

Present and flow values

- Consider the present value

$$v(t) = \int_t^\infty e^{-\rho(s-t)}\,\pi(s)\,ds, \qquad t \geq 0$$

- Differentiating with respect to time t

$$\dot{v}(t) = \int_t^\infty \left[\rho e^{-\rho(s-t)}\,\pi(s)\right] ds \;+\; e^{-\rho(s-t)}\,\pi(s)\Big|_{s=t}(-1)$$

$$= \rho v(t) - \pi(t)$$

- Commonly written

$$\rho v(t) = \pi(t) + \dot{v}(t)$$

and in steady state, naturally $v = \pi/\rho$

Deterministic control

- Now consider the control problem

$$v(x(t),t) = \max_{u(\cdot)} \int_t^{\infty} e^{-\rho(s-t)}\,\pi(x(s),u(s))\,ds, \qquad t \geq 0$$

with state variable x, control u, subject to the law of motion

$$\dot{x}(t) = g(x(t),u(t)), \qquad x(0) = x_0 \text{ given}$$

- Value function satisfies the continuous time Bellman equation

$$\rho v(x,t) = \max_u \left[\pi(x,u) + \frac{\partial v}{\partial x} g(x,u)\right] + \frac{\partial v}{\partial t}$$

where the maximand is the Hamiltonian function

$$H(x,u,\lambda) = \pi(x,u) + \lambda g(x,u)$$

with costate variable $\lambda = \partial v / \partial x$

Heuristic derivation of the Bellman equation

- Discrete time analogue with period length $\Delta t > 0$, $t' = t + \Delta t$, $x' = x + \Delta x$ satisfies

$$v(x,t) = \max_u \left[\pi(x,u)\,\Delta t + \frac{1}{1+\rho\Delta t} v(x',t') \right]$$

- Multiplying both sides by $(1+\rho\Delta t)$ and subtracting $v(x,t)$

$$\rho v(x,t)\Delta t = \max_u \left[\pi(x,u)\,(1+\rho\Delta t)\Delta t + v(x',t') - v(x,t) \right]$$

- Write out the change in the value function as follows

$$v(x',t') - v(x,t) = v(x',t') - v(x,t') + v(x,t') - v(x,t)$$

$$= \frac{v(x',t') - v(x,t')}{\Delta x}\Delta x + \frac{v(x,t') - v(x,t)}{\Delta t}\Delta t$$

Heuristic derivation of the Bellman equation

- Now use the state transition equation $\Delta x = g(x, u)\Delta t$ to write

$$v(x', t') - v(x, t) = \frac{v(x', t') - v(x, t')}{\Delta x} g(x, u)\Delta t + \frac{v(x, t') - v(x, t)}{\Delta t} \Delta t$$

- Plug change in value function back into Bellman equation

$$\rho v(x, t)\Delta t = \max_u \left[\pi(x, u)\,(1 + \rho \Delta t)\Delta t + \frac{v(x', t') - v(x, t')}{\Delta x} g(x, u)\Delta t \right] + \frac{v(x, t') - v(x, t)}{\Delta t} \Delta t$$

- Divide both sides by $\Delta t > 0$ and take limit as $\Delta t \to 0$ to get

$$\rho v(x, t) = \max_u \left[\pi(x, u) + \frac{\partial v}{\partial x} g(x, u) \right] + \frac{\partial v}{\partial t}$$

Poisson process

- Continuous time stochastic process $x(t)$ with $x(0) = 0$ and

  (i) stationary independent increments
  (ii) increments have Poisson distribution with intensity $\lambda > 0$

- Increments $x(t') - x(t)$ are independent r.v.'s, for all t, t'

- Increments $\Delta x = x(t + \Delta t) - x(t)$ have Poisson distribution

$$\text{Prob}[\Delta x = k] = \frac{(\lambda \Delta t)^k e^{-\lambda \Delta t}}{k!}$$

  (depends only on period length $\Delta t > 0$, independent of actual t)

- Sample paths are discontinuous, a 'counting process'

Properties of Poisson process

- Write the distribution of the increment from date s to $s+t$

$$\text{Prob}[\,\{x(t+s)-x(s)\}\;=k]=\frac{(\lambda t)^k e^{-\lambda t}}{k!}$$

that is, a Poisson distribution with parameter λt

- In particular, taking $s=0$ (and using $x(0)=0$) we simply have

$$\text{Prob}[x(t)=k]=\frac{(\lambda t)^k e^{-\lambda t}}{k!}$$

- So, using standard properties of the Poisson distribution, the process $x(t)$ has moments

$$\mathbb{E}[x]=\text{Var}[x]=\lambda t$$

Properties of Poisson process

- Probability that $\Delta x = 1$ over period of length Δt is

$$\text{Prob}[\Delta x = 1] = (\lambda \Delta t) e^{-\lambda \Delta t}$$

- Probability process does not change, $\Delta x = 0$, over same period

$$\text{Prob}[\Delta x = 0] = e^{-\lambda \Delta t}$$

(note link to exponential distribution...)

- These can be written

$$\text{Prob}[\Delta x = 1] = \lambda \Delta t + o(\Delta t)$$
$$\text{Prob}[\Delta x = 0] = 1 - \lambda \Delta t + o(\Delta t)$$

where $o(\Delta t)/\Delta t \to 0$ as $\Delta t \to 0$ and moreover where

$$\text{Prob}[\Delta x > 1] = o(\Delta t)$$

Bellman equation when state is Poisson

- Consider the value function

$$v(x(t),t) = \mathbb{E}\Big[\int_t^{\infty} e^{-\rho(s-t)}\pi(x(s))\,ds \,\Big|\, x(t)\Big], \qquad t \geq 0$$

  where $x(t)$ follows a Poisson process with intensity λ

- What is the Bellman equation for this problem?

- Discrete time analogue with period length $\Delta t > 0$, $t' = t + \Delta t$, $x' = x + \Delta x$ satisfies

$$v(x,t) = \pi(x)\Delta t + \frac{1}{1+\rho\Delta t}\mathbb{E}[v(x',t')\,|\,x]$$

Bellman equation when state is Poisson

- Multiplying both sides by $(1+\rho\Delta t)$ and subtracting $v(x,t)$ gives

$$\rho v(x,t)\Delta t = \pi(x)\Delta t(1+\rho\Delta t) + \mathbb{E}[v(x',t') - v(x,t)\,|\,x]$$

- Again write out the change in the value function

$$v(x',t') - v(x,t) = v(x',t') - v(x,t') + v(x,t') - v(x,t)$$

$$= v(x',t') - v(x,t') + \frac{v(x,t') - v(x,t)}{\Delta t}\Delta t$$

- The latter change is *deterministic* and can be pulled outside of the expectation, so

$$\rho v(x,t)\Delta t = \pi(x)\Delta t(1+\rho\Delta t) + \mathbb{E}[v(x',t') - v(x,t')\,|\,x] + \frac{v(x,t') - v(x,t)}{\Delta t}\Delta t$$

Bellman equation when state is Poisson

- Now write out the expectation

$$\mathbb{E}[v(x',t') - v(x,t') \,|\, x] = \sum_{\Delta x=0}^{\infty} [v(x+\Delta x, t') - v(x,t')]\mathrm{Prob}[\Delta x]$$

$$\begin{aligned} = \quad & [v(x+0,t') - v(x,t')]\mathrm{Prob}[\Delta x = 0] \\ & + [v(x+1,t') - v(x,t')]\mathrm{Prob}[\Delta x = 1] \\ & + [v(x+2,t') - v(x,t')]\mathrm{Prob}[\Delta x = 2] \\ & + \ldots \end{aligned}$$

- Hence for the Poisson process

$$\mathbb{E}[v(x',t') - v(x,t') \,|\, x] = [v(x+1, t+\Delta t) - v(x, t+\Delta t)]\lambda \Delta t + o(\Delta t)$$

Bellman equation when state is Poisson

- Plug this back into the Bellman equation

$$\begin{aligned}\rho v(x,t)\Delta t = {} & \pi(x)\Delta t(1+\rho\Delta t) \\ & + [v(x+1,t+\Delta t) - v(x,t+\Delta t)]\lambda\Delta t + o(\Delta t) \\ & + \frac{v(x,t') - v(x,t)}{\Delta t}\Delta t\end{aligned}$$

- Divide both sides by $\Delta t > 0$ and take limit as $\Delta t \to 0$ to get

$$\rho v(x,t) = \pi(x) + \lambda(v(x+1,t) - v(x,t)) + \frac{\partial v}{\partial t}$$

Application to quality ladder model

- Let x denote number of quality improvements that have occurred

- Let $V(x,t)$ denote value function of an incumbent firm

- For an incumbent, x matters *only if an entrant makes an innovation* — in which case it loses its whole market

- In short, write $v(t) := V(x,t)$ and $\pi(x) = \pi$ so long as x remains unchanged, write $V(x+1,t) = 0$ if entrant innovates, etc

- Then we have the Bellman equation from the Lecture 5 notes

$$\rho v(t) = \pi - \lambda v(t) + \dot{v}(t)$$

and in steady-state, naturally $v = \pi/(\rho + \lambda)$, i.e., profits discounted at the *risk-adjusted* rate $\rho + \lambda$