PhD Topics in Macroeconomics

Lecture 3: firm dynamics, part three

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This lecture

1- Background for discrete state dynamic programming

- Markov chains, review
- numerical integration, review
- using Markov chains to approximate processes with continuous support

2- Solving the Hopenhayn model by discrete state dynamic programming

3- Results from simple numerical example

Markov chains

- A finite *Markov chain* is a triple (x , P , f_0) where

  x is an n-vector listing the possible **states** (outcomes) of the chain

  P is an $n \times n$ **probability transition matrix**

  f_0 is an n-vector recording the **initial distribution** over the states

- Restrictions

$$0 \leq p_{ij} \leq 1, \qquad \text{and} \qquad \sum_{j=1}^{n} p_{ij} = 1 \qquad \text{for all } i = 1, ..., n$$

$$0 \leq f_{0,i} \leq 1, \qquad \text{and} \qquad \sum_{i=1}^{n} f_{0,i} = 1$$

Interpretation

- Consider *stochastic process* $\{X_t\}_{t=0}^{\infty}$ induced by a Markov chain
- A realization of X_t takes on the value of one of the states in x
- Elements p_{ij} of the transition matrix P then

$$p_{ij} = \text{Prob}[X_{t+1} = x_j \,|\, X_t = x_i]$$

- Elements $f_{0,i}$ of the initial distribution

$$f_{0,i} = \text{Prob}[X_0 = x_i]$$

Transitions

- Let the vector f_t be the distribution at time t, with elements

$$f_{t,i} = \text{Prob}[X_t = x_i]$$

- Using the transition probabilities gives

$$f_{1,i} = \sum_{j=1}^{n} \text{Prob}[X_1 = x_i \,|\, X_0 = x_j] \, \text{Prob}[X_0 = x_j]$$

$$\vdots$$

$$f_{t+1,i} = \sum_{j=1}^{n} \text{Prob}[X_{t+1} = x_i \,|\, X_t = x_j] \, \text{Prob}[X_t = x_j]$$

Transitions

- Collecting these together in matrix notation, we see that

$$
\begin{aligned}
f_1 &= (P')f_0 \\
&\vdots \\
f_{t+1} &= (P')f_t, \qquad t = 0, 1, \ldots
\end{aligned}
$$

  where P' denotes the *transpose* of P

- Evolves according to a *deterministic* difference equation

- Iterating forward from date $t = 0$ we have

$$f_t = (P')^t f_0$$

Stationary distributions

- Stationary distribution $\bar{f}$ of Markov chain satisfies

$$\bar{f} = P'\bar{f}$$

(i.e., a *steady state* of the difference equation $f_{t+1} = (P')f_t$)

- Writing this as

$$(I - P')\bar{f} = 0$$

we see $\bar{f}$ is an *eigenvector* of P' associated with a *unit-eigenvalue*

- Requirement that $\sum_i \bar{f}_i = 1$ is a normalization of the eigenvector

Uniqueness and stability (sketch)

- Generally P' has n eigenvalues
- Since P is a transition matrix, P' has *at least one* unit-eigenvalue
- But may have *multiple* unit-eigenvalues, hence multiple stationary distributions
- Moreover even if there is a unique stationary distribution, iterates $f_{t+1} = (P')f_t$ may not converge to it
- A *sufficient condition* for a unique stable stationary distribution is that $0 < p_{ij} < 1$ for all i, j

2×2 example

- Consider two state Markov chain with transition matrix

$$P = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}$$

- Stationary distribution solves (note the transpose!)

$$\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1-p & q \\ p & 1-q \end{pmatrix} \right] \begin{pmatrix} \bar{f}_1 \\ \bar{f}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- Gives

$$\begin{pmatrix} \bar{f}_1 \\ \bar{f}_2 \end{pmatrix} = \begin{pmatrix} \frac{q}{p+q} \\ \frac{p}{p+q} \end{pmatrix}$$

(e.g., $q \to 0$ makes state 2 *absorbing* and state 1 *transient*, etc)

Continuous support

- Suppose a realization of X_t is drawn from a continuous distribution with PDF $f_t(x)$

- Intuitively

$$f_{t+1}(x') = \int p(x' \,|\, x) f_t(x) \, dx$$

where $p(x' \,|\, x)$ is density for $X_{t+1} = x'$ conditional on $X_t = x$

- Analogous theory of uniqueness, stability etc for stationary distributions $\bar{f}(x)$

AR(1) example

- Suppose $\{X_t\}_{t=0}^{\infty}$ is a linear Gaussian AR(1) process

$$X_{t+1} = (1 - \rho)\mu + \rho X_t + \sigma Z_{t+1}, \qquad Z_{t+1} \sim \text{IID } N(0, 1)$$

- Then

$$p(x' \mid x) = \frac{1}{\sigma} \phi \left(\frac{x' - (1 - \rho)\mu - \rho x}{\sigma} \right)$$

where $\phi(z)$ is the PDF of the standard normal distribution

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

AR(1) example

- If $|\rho| < 1$, then a unique, stable stationary distribution with PDF

$$\bar{f}(x) = \frac{1}{\bar{\sigma}} \phi \left(\frac{x - \mu}{\bar{\sigma}} \right)$$

where

$$\bar{\sigma} = \frac{\sigma}{\sqrt{1 - \rho^2}}$$

Numerical integration (quadrature)

- Consider integral of a function $f(x)$ against weights $w(x)$

$$I = \int f(x) w(x)\, dx$$

- Often not possible to calculate the integral exactly

- Can approximate the integral value by choosing an appropriate set of *quadrature nodes* x_i and *weights* w_i so that

$$I = \int f(x) w(x)\, dx \approx \sum_{i=1}^{n} f(x_i)\, w_i$$

- Various procedures for choosing nodes x_i and weights w_i (Newton-Cotes, Gaussian, Monte Carlo, etc)

Gaussian quadrature

- Choose nodes x_i and weights w_i to satisfy $2n$ *'moment conditions'*

$$\int x^k w(x)\,dx = \sum_{i=1}^{n} x_i^k w_i, \qquad k = 0, ..., 2n-1$$

(2n nonlinear equations in 2n unknowns, nontrivial but standard routines exist)

- **Note:** if X is a continuous random variable with PDF $w(x)$ then Gaussian quadrature "discretizes" X, replacing it with n discrete points x_i and a PMF w_i on those discrete points

- The discretized version approximates the continuous version in the sense that the first $2n$ moments are the same

Gaussian quadrature, 3-point example

- Suppose $w(x) = \phi(x)$, the standard normal density

- Choose 3 nodes and 3 weights to satisfy 6 moments

$$\mathbb{E}[x^0] = 1, \quad \mathbb{E}[x^1] = 0, \quad \mathbb{E}[x^2] = 1,$$
$$\mathbb{E}[x^3] = 0, \quad \mathbb{E}[x^4] = 3, \quad \mathbb{E}[x^5] = 0$$

- Solution to system of 6 equations in 6 unknowns is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -\sqrt{3} \\ 0 \\ +\sqrt{3} \end{pmatrix}, \qquad \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 1/6 \\ 2/3 \\ 1/6 \end{pmatrix}$$

Tauchen/Hussey (1991) approximation

- Similarly, can use quadrature to obtain discrete Markov chain approximation to process with continuous support

- Recall density for $X_{t+1} = x'$ conditional on $X_t = x$

$$p(x' \,|\, x)$$

- Discretize support of X to n quadrature nodes x_i and replace $p(x' \,|\, x)$ by $n \times n$ matrix of transition probabilities

$$p_{ij} = \frac{p(x_j \,|\, x_i)\frac{w_j}{\omega(x_j)}}{\sum_{j'=1}^{n} p(x_{j'} \,|\, x_i)\frac{w_{j'}}{\omega(x_{j'})}}, \qquad i, j = 1, ..., n$$

where w_i are the weights for x_i and $\omega(x)$ is a *'regularity function'* that controls the quality of the approximation to higher moments

Tauchen/Hussey (1991) example

- Suppose we want to approximate AR(1) with Markov chain

$$p(x' \mid x) = \frac{1}{\sigma}\phi\left(\frac{x' - (1-\rho)\mu - \rho x}{\sigma}\right)$$

- Lookup quadrature nodes x_i, weights w_i for normal $N(\mu, \hat{\sigma}^2)$. Set regularity function to

$$\omega(x) = \frac{1}{\hat{\sigma}}\phi\left(\frac{x-\mu}{\hat{\sigma}}\right)$$

- Tauchen/Hussey (1991) advocate $\hat{\sigma} = \sigma$ (innovation std dev). But Floden (2008) advocates that for highly persistent processes

$$\hat{\sigma} = \theta\sigma + (1-\theta)\bar{\sigma}, \qquad \theta = 1/2 + \rho/4$$

($\rho \approx 1 \Rightarrow$ more weight in tails, better match conditional variance)

Solving the Hopenhayn model

- Back to firm dynamics ...

- Suppose productivity follows n-state Markov chain on a_i with transition probabilities f_{ij}

- Given price p, value function is a n-vector with elements $v_i(p)$, i.e,

$$v_i(p) := v(a_i, p), \quad \pi_i(p) := \pi(a_i, p), \quad y_i(p) := y(a_i, p), \quad \text{etc}$$

- Bellman equation for incumbent firm is then

$$v_i(p) = \pi_i(p) + \beta \max \left[0 \, , \, \sum_{j=1}^{n} v_j(p) \, f_{ij} \right]$$

Value function iteration

- Stacking the values into a vector v, this is a mapping of the form

  $$v = T(v, p)$$

  For given p, find $v^*(p)$ that solves this fixed point problem

- Iterating on T from some initial guess v^0 gives

  $$T(v^k, p) = v^{k+1} \to v^*(p) \qquad \text{as } k \to \infty$$

  Note: will work because T is a *contraction mapping* (it satisfies *Blackwell's sufficient conditions*: monotonicity and discounting)

- In practice, iterate on T until

  $$\|v^{k+1} - v^k\| < \epsilon$$

  for some pre-specified tolerance $\epsilon > 0$, say $\epsilon = 10^{-6}$

Free entry

- Let $g_i = g(a_i)$ denote PMF of initial distribution over nodes a_i

- Given $v(p)$ that solves incumbent's problem, free entry condition is

$$v^e(p) := \beta \sum_{i=1}^{n} v_i(p) g_i = k_e$$

  whenever there is positive entry, $m > 0$ (for some parameter values, may have $m = 0$ in which case $v^e(p) < k_e$, see below)

- Easy to show that $v^e(0) < 0$ and $v^e(p)$ monotone increasing in p, so interior solutions (with $m > 0$) can be found by *bisection*

- Intuitively, if $v^e(p) > k_e$ then reduce price to discourage entry but if $v^e(p) < k_e$ then increase price to encourage entry

Exit decisions

- Given p^*, we know incumbent value function $v(p^*)$

- Exit threshold $a(p^*)$ then found from

$$a(p^*) = a_{i^*}, \qquad i^* := \min_i \left[\sum_{j=1}^{n} v_j(p^*)\, f_{ij} \geq 0 \right]$$

  All firms with $a_i < a(p^*)$ *exit*, all firms with $a_i \geq a(p^*)$ *continue*

- Collect exit decisions into a vector $x(p^*)$ with elements

$$\begin{aligned} x_i(p^*) &= 1 \qquad \text{if } a_i < a(p^*) \\ x_i(p^*) &= 0 \qquad \text{if } a_i \geq a(p^*) \end{aligned}$$

Distribution dynamics

- Let $\mu_{it} = \mu_t(a_i)$ denote mass of firms with productivity a_i at t

- Vector μ_t evolves according to difference equation

$$\mu_{t+1} = \Psi(p^*)\mu_t + m\, g, \qquad t = 0, 1, ...$$

where $n \times n$ coefficient matrix $\Psi(p^*)$ has elements

$$\psi_{ij}(p^*) = (1 - x_j(p^*))\, f_{ji}, \qquad i, j = 1, ..., n$$

- Mass of firms at node a_i at $t+1$ depends on transition probabilities and exit decisions of incumbents at t and flow of new entrants

- Stationary distribution

$$\mu = m(I - \Psi(p^*))^{-1} g =: \mu(m, p^*)$$

for some m yet to be determined

Market clearing

- Industry demand curve, exogenous $D(p)$

- Industry supply curve, endogenous

$$Y(m,p) = \sum_{i=1}^{n} y_i(p)\mu_i(m,p)$$

- We have solved for p^* from free entry condition (supposing $m > 0$). So now want to find measure of entrants m^* that solves

$$Y(m,p^*) = \sum_{i=1}^{n} y_i(p^*)\mu_i(m,p^*) = D(p^*)$$

- **Trick:** $\mu(m,p)$ is linear in m, so write $\mu(m,p) = m \times \mu(1,p)$ and solve for m^* as

$$m^* = \frac{D(p^*)}{Y(1,p^*)} = \frac{D(p^*)}{\sum_{i=1}^{n} y_i(p^*)\mu(1,p^*)}$$

Aside: corner solutions

- What if $m^* = 0$? (*no entry*)

- Then, in stationary equilibrium, can also be *no exit*

- Stationary distribution of firms just given by stationary distribution of Markov chain

$$\mu_i = \bar{f}_i$$

- Then market clears if

$$Y(p) = \sum_{i=1}^{n} y_i(p)\bar{f}_i = D(p)$$

Solve for p^* (no longer use free-entry condition to determine p^*)

Numerical example

- Suppose preferences and technology

$$y = an^{\alpha}, \qquad D(p) = \bar{D}/p$$

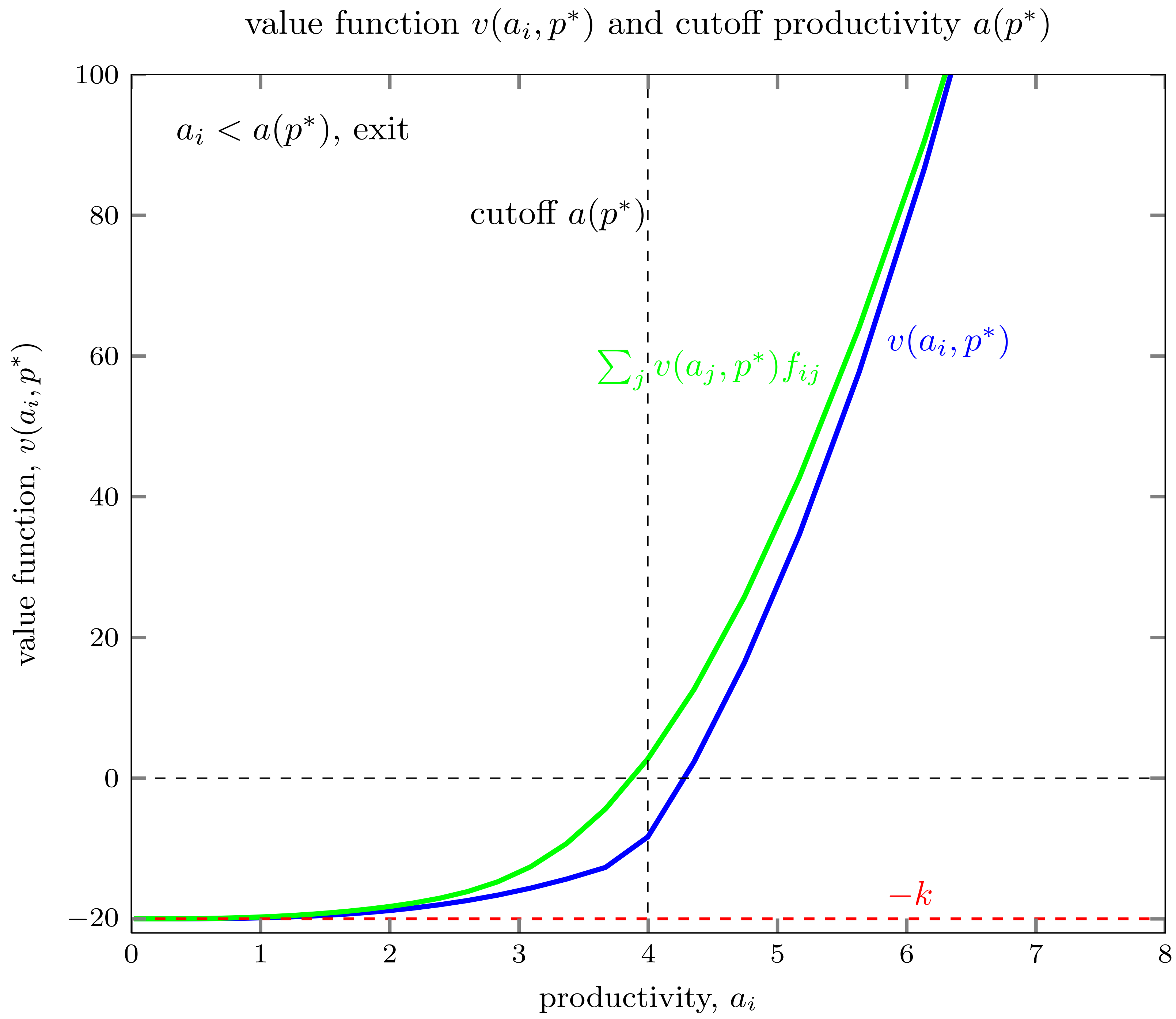
- And that firm productivity follows AR(1) in logs

$$\log a_{t+1} = (1-\rho)\log \bar{a} + \rho \log a_t + \sigma \varepsilon_{t+1}$$

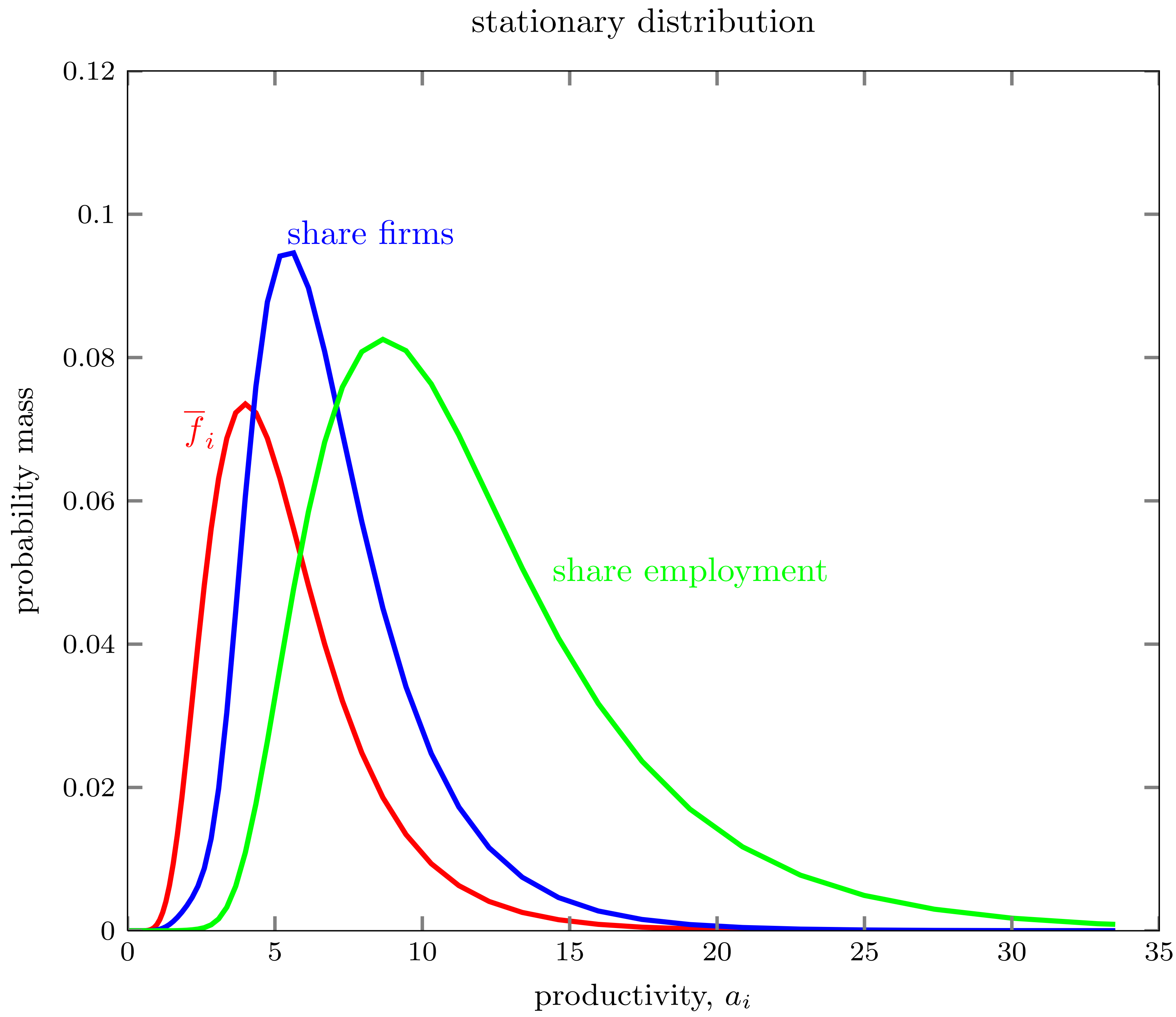
- Parameter values (period length 5 years, more on this next class)

$$\alpha = 2/3, \quad \beta = 0.80, \quad k = 20, \quad k_e = 40$$
$$\log \bar{a} = 1.39, \quad \sigma = 0.20, \quad \rho = 0.9, \quad \bar{D} = 100$$

- Approximate AR(1) with Markov chain on 101 nodes



value function $v(a_i, p^*)$ and cutoff productivity $a(p^*)$
$a_i < a(p^*)$, exit
cutoff $a(p^*)$
$\sum_j v(a_j, p^*) f_{ij}$
$v(a_i, p^*)$
$-k$
value function, $v(a_i, p^*)$
productivity, a_i
100
80
60
40
20
0
-20
0
1
2
3
4
5
6
7
8


stationary distribution
0.12
0.1
0.08
0.06
0.04
0.02
0
probability mass
share firms
$\overline{f}_i$
share employment
0
5
10
15
20
25
30
35
productivity, a_i

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Hopenhayn Example

price                                       =     1.0
aggregate output                            =   100.0
aggregate productivity                      =     1.5
aggregate employment, production            =    66.7
aggregate employment, overhead              =    12.7
aggregate profit                            =    20.6

exit/entry rate                             =   0.139
gross firm turnover rate                    =   0.278

average firm size                           =   66.67

size                   <20    <50   <100   <500   rest

share firms           0.20   0.26   0.27   0.25   0.03
share employment      0.02   0.08   0.17   0.47   0.26
```

Next

- Firm dynamics: basic models, part four

- General equilibrium and a substantive application

- Reading:

  - Hopenhayn and Rogerson (1993): Job turnover and policy evaluation: A general equilibrium analysis," *Journal of Political Economy.*

- Nonconvex adjustment costs, a firm's lagged employment is an endogenous state variable