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Chris Edmond

Endogenous asset market segmentation

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This note outlines the main details of Alvarez, Atkeson and Kehoe (2002). This is a compete markets cash-in-advance economy along the lines of Note 1 but where households must pay a fixed cost γ to transfer money between the asset market and the goods market.

1 Environment

Let there be countable dates, $t = 0, 1, 2....$ There is a single *aggregate* history of money growth shocks $\mu^t = (\mu_0, ..., \mu_t)$ with density $g_t(\mu^t)$ as of period zero. The realization μ_t is known at the beginning of period t.

Agents and endowments. There is a continuum of ex ante identical households, a large number of perfectly competitive financial intermediaries, and a government. As in a typical cash-in-advance economy each household splits into a worker and a shopper each period. Each period each household receives an idiosyncratic endowment $y \in [0, \infty)$ that is IID in the population and IID over time with density $f(y)$ and distribution F. Therefore (i) an *idiosyncratic endowment* history $y^t = (y_0, ..., y_t)$ has density $f_t(y^t) := f(y_0)f(y_1)\ldots f(y_t)$ as of period zero, and (ii) the aggregate endowment $Y := \int_0^\infty yf(y)dy$ is constant each period.

Note: the realization y_t is *not known* at the beginning of period t. Instead this only becomes known to a household when its worker fetches the endowment part way through the period. Household choices at the beginning of period t are therefore a function of (μ^t, y^{t-1}) not of (μ^t, y^t).

Markets and timing. There are two markets, an asset market and a goods markets. At the beginning of each period, households trade money and nominal bonds with perfectly competitive financial intermediaries in the asset market and the government injects money into the asset market via open market operations. For each period $t \geq 1$, after the asset market closes, the goods market opens. In the goods market, the household's shopper uses money to make purchases of the consumption good and the household's worker sells their endowment for money. The goods market is closed at date $t = 0$ so as to ensure that all households are ex ante identical.

Government flow budget constraint. The government begins each period $t \geq 1$ with outstanding liabilities $\hat{B}_t(\mu^{t-1}, \mu_t)$ which can be covered by printing money $\hat{M}_t(\mu^t)$ or by selling more state contingent bonds

$$\hat{B}_t(\mu^{t-1}, \mu_t) \leq \hat{M}_t(\mu^t) - \hat{M}_{t-1}(\mu^{t-1}) + \int_{\mu'} q_t(\mu^t, \mu')\hat{B}_{t+1}(\mu^t, \mu')d\mu'$$

Here $q_t(\mu^t, \mu')$ denotes the price of a nominal state contingent bond that pays out if μ' realizes at date $t+1$.

Financial intermediaries. Each period $t \geq 1$ households purchase nominal bonds from perfectly competitive financial intermediaries that are contingent both to the aggregate μ^t and to the household's own idiosyncratic endowment. Let $B_{t+1}(\mu^t, \mu', y^{t-1}, y)$ denote bonds carried into period $t+1$ that pay one dollar in the asset market if y is realized later this period during the goods market and μ' is realized at the beginning of period $t+1$. With apologies for the slight abuse of notation, let $q_t(\mu^t, \mu', y^{t-1}, y)$ denote the price of this nominal state continent bond.

Each period $t \geq 1$ intermediaries sell idiosyncratic bonds to the population of households of type $y^t = (y^{t-1}, y_t)$ and buy aggregate bonds from the government to maximize profits

$$\int_{\mu'} \left[\int_{y^t} q_t(\mu^t, \mu', y^t)B_{t+1}(\mu^t, \mu', y^t)f_t(y^t)dy^t - q_t(\mu^t, \mu')\hat{B}_{t+1}(\mu^t, \mu') \right] d\mu' \tag{1}$$

subject to the constraint that dollars promised to households equals dollar promised by the government

$$\int_{y^t} B_{t+1}(\mu^t, \mu', y^t)f_t(y^t)dy^t = \hat{B}_{t+1}(\mu^t, \mu') \tag{2}$$

Zero profits then implies that the relationship between the price of idiosyncratic bonds and aggregate bonds can be written

$$q_t(\mu^t, \mu', y) = q_t(\mu^t, \mu')f(y) \tag{3}$$

with further apologies for the abuse of notation. Notice that this price does not depend on y^{t-1}. This is because y_t is IID over time and so y^{t-1} provides no information about the likelihood of a particular y_t realization.

Household flow asset market constraints. In the asset market each period $t \geq 1$ households have payments from bond holdings and they can use these funds to buy more bonds from the financial intermediaries (re-investing the bonds) or make transfers to the goods market

$$P_t(\mu^t)[x_t(\mu^t, y^{t-1}) + \gamma]z_t(\mu^t, y^{t-1}) + \int_{\mu'}\int_y q_t(\mu^t, \mu')f(y)B_{t+1}(\mu^t, \mu', y^{t-1}, y)d\mu' dy \leq B_t(\mu^{t-1}, \mu_t, y^{t-1}) \tag{4}$$

where $P_t(\mu^t)$ denotes the aggregate price level, $x_t(\mu^t, y^{t-1})$ denotes the real value of a transfer from the asset market to the goods market and where $z_t(\mu^t, y^{t-1}) \in \{0, 1\}$ is an indicator function that equals 0 if there is no transfer and 1 if there is a transfer. The fixed cost of the transfer is γ units of real resources paid in money in the asset market.

Households are assumed not to hold money over in the asset market (i.e., that the shocks are such that money is dominated in rate of return by bonds in the asset market).

Cash-in-advance constraints. In the goods market each period $t \geq 1$ households begin with the money brought home by the worker at the end of the previous period, $P_{t-1}(\mu^{t-1})y_{t-1}$ or equivalently real balances

$$m_t(\mu^t, y^{t-1}) := \frac{P_{t-1}(\mu^{t-1})}{P_t(\mu^t)} y_{t-1}$$

If a transfer was made from the asset market, then the household has an additional $x_t(\mu^t, y^{t-1})$ real balances. The cash-in-advance constraint is therefore, in real terms,

$$c_t(\mu^t, y^{t-1}) = m_t(\mu^t, y^{t-1}) + x_t(\mu^t, y^{t-1}) z_t(\mu^t, y^{t-1}) \tag{5}$$

The cash-in-advance constraint is assumed to be binding.

Household intertemporal budget constraint. The household's sequence of flow asset market constraints can be iterated forward to get a single intertemporal constraint

$$\sum_{t=0}^{\infty} \int_{\mu^t} \int_{y^{t-1}} Q_t(\mu^t) P_t(\mu^t)[x_t(\mu^t, y^{t-1}) + \gamma] z_t(\mu^t, y^{t-1}) f_{t-1}(y^{t-1}) d\mu^t dy^{t-1} \leq \overline{B} \tag{6}$$

where $Q_t(\mu^t)$ denotes the price at date zero of a bond that pays one dollar into the asset market at date t after history μ^t and where $\overline{B}$ denotes each households endowment of initial bonds in the asset market at date zero. Notice that all households have the same intertemporal budget constraint.

Household preferences. Each household has the same time and state separable expected utility function

$$\sum_{t=0}^{\infty} \int_{\mu^t} \int_{y^{t-1}} \beta^t U[c_t(\mu^t, y^{t-1})] g_t(\mu^t) f_{t-1}(y^{t-1}) d\mu^t dy^{t-1} \tag{7}$$

which they maximize subject to the single intertemporal budget constraint and the sequence of cash-in-advance constraints.

Market clearing. Since γ units of real goods are used up with each transfer, the goods market clears when

$$\int_{y^{t-1}} [c_t(\mu^t, y^{t-1}) + \gamma z_t(\mu^t, y^{t-1})] f_{t-1}(y^{t-1}) dy^{t-1} = Y \tag{8}$$

and the asset market clears when

$$\int_{y^{t-1}} \{m_t(\mu^t, y^{t-1}) + [x_t(\mu^t, y^{t-1}) + \gamma] z_t(\mu^t, y^{t-1})\} f_{t-1}(y^{t-1}) dy^{t-1} = \frac{\hat{M}_t(\mu^t)}{P_t(\mu^t)} \tag{9}$$

2 Optimization

Let $\lambda \geq 0$ denote the Lagrange multiplier on the household's single intertemporal budget constraint. Since households are ex ante identical, this multiplier is the same for every household. Let $\nu_t(\mu^t, y^{t-1}) \geq 0$ denote the multipliers on the cash-in-advance constraints. The Lagrangian for the household's problem can be written

$$\begin{aligned} L &= \sum_{t=0}^{\infty} \int_{\mu^t} \int_{y^{t-1}} \beta^t U[c_t(\mu^t, y^{t-1})] g_t(\mu^t) f_{t-1}(y^{t-1}) d\mu^t dy^{t-1} \\ &+ \lambda \left\{ \overline{B} - \sum_{t=0}^{\infty} \int_{\mu^t} \int_{y^{t-1}} Q_t(\mu^t) P_t(\mu^t) [x_t(\mu^t, y^{t-1}) + \gamma] z_t(\mu^t, y^{t-1}) f_{t-1}(y^{t-1}) d\mu^t dy^{t-1} \right\} \\ &+ \sum_{t=0}^{\infty} \int_{\mu^t} \int_{y^{t-1}} \nu_t(\mu^t, y^{t-1}) [m_t(\mu^t, y^{t-1}) + x_t(\mu^t, y^{t-1}) z_t(\mu^t, y^{t-1}) - c_t(\mu^t, y^{t-1})] \end{aligned}$$

The key first order conditions for consumption and transfers are

$$c_t(\mu^t, y^{t-1}) \quad : \quad \beta^t U'[c_t(\mu^t, y^{t-1})] g_t(\mu^t) f_{t-1}(y^{t-1}) = \nu_t(\mu^t, y^{t-1}) \tag{10}$$

$$x_t(\mu^t, y^{t-1}) \quad : \quad \lambda Q_t(\mu^t) P_t(\mu^t) z_t(\mu^t, y^{t-1}) f_{t-1}(y^{t-1}) = \nu_t(\mu^t, y^{t-1}) z_t(\mu^t, y^{t-1}) \tag{11}$$

If there is a transfer from the asset market to the goods market, $z_t(\mu^t, y^{t-1}) = 1$ then we can combine these equations to conclude

$$\beta^t U'[c_t(\mu^t, y^{t-1})] g_t(\mu^t) = \lambda Q_t(\mu^t) P_t(\mu^t)$$

This implies that for all households that make a transfer, consumption is $c_t(\mu^t, y^{t-1}) = \bar{c}_t(\mu^t)$ for some function $\bar{c}_t(\mu^t)$ to be determined. In particular, this consumption level is independent of the idiosyncratic endowments of participating households. If there is no transfer from the asset market to the goods market, $z_t(\mu^t, y^{t-1}) = 0$, then consumption is just equal to real balances $c_t(\mu^t, y^{t-1}) = m_t(\mu^t, y^{t-1})$ which does vary with the household's idiosyncratic endowment. In short, households that make a transfer share their risks with all other households that make a transfer while households that do not make a transfer do not get to share risks. We now need to determine when in fact $z_t(\mu^t, y^{t-1}) = 0$ and when $z_t(\mu^t, y^{t-1}) = 1$.

Taking $\bar{c}_t(\mu^t)$ as given we now need to choose $z_t(\mu^t, y^{t-1})$ to maximize

$$\sum_{t=0}^{\infty} \int_{\mu^t} \int_{y^{t-1}} \beta^t \left\{ z_t(\mu^t, y^{t-1}) U[\bar{c}_t(\mu^t)] + (1 - z_t(\mu^t, y^{t-1})) U[m_t(\mu^t, y^{t-1})] \right\} g_t(\mu^t) f_{t-1}(y^{t-1}) d\mu^t dy^{t-1}$$

subject to the single intertemporal constraint

$$\sum_{t=0}^{\infty} \int_{\mu^t} \int_{y^{t-1}} Q_t(\mu^t) P_t(\mu^t) [\bar{c}_t(\mu^t) - m_t(\mu^t, y^{t-1}) + \gamma] z_t(\mu^t, y^{t-1}) f_{t-1}(y^{t-1}) d\mu^t dy^{t-1} \leq \overline{B}$$

The 'kernel' of the associated Lagrangian is

$$\begin{aligned} & \beta^t \left\{ z_t(\mu^t, y^{t-1}) U[\bar{c}_t(\mu^t)] + (1 - z_t(\mu^t, y^{t-1})) U[m_t(\mu^t, y^{t-1})] \right\} g_t(\mu^t) \\ - \quad & \lambda Q_t(\mu^t) P_t(\mu^t) [\bar{c}_t(\mu^t) - m_t(\mu^t, y^{t-1}) + \gamma] z_t(\mu^t, y^{t-1}) \end{aligned}$$

Hence the contribution to the Lagrangian when there is no transfer $z_t(\mu^t, y^{t-1}) = 0$ is just

$$\beta^t U[m_t(\mu^t, y^{t-1})] g_t(\mu^t) \tag{12}$$

while the contribution to the Lagrangian when there is a transfer $z_t(\mu^t, y^{t-1}) = 1$ is

$$\beta^t U[\bar{c}_t(\mu^t)] g_t(\mu^t) - \lambda Q_t(\mu^t) P_t(\mu^t) [\bar{c}_t(\mu^t) - m_t(\mu^t, y^{t-1}) + \gamma] \tag{13}$$

and using the first order condition for choice of consumption when $z_t(\mu^t, y^{t-1}) = 1$ we have $\beta^t U'[\bar{c}_t(\mu^t)] g_t(\mu^t) = \lambda Q_t(\mu^t) P_t(\mu^t)$ so that this contribution to the Lagrangian can be written

$$\beta^t U[\bar{c}_t(\mu^t)] g_t(\mu^t) - \beta^t U'[\bar{c}_t(\mu^t)] [\bar{c}_t(\mu^t) - m_t(\mu^t, y^{t-1}) + \gamma] g_t(\mu^t) \tag{14}$$

Putting these two contributions together, we conclude that the household makes a transfer if and only if

$$U[\bar{c}_t(\mu^t)] - U'[\bar{c}_t(\mu^t)] [\bar{c}_t(\mu^t) - m_t(\mu^t, y^{t-1}) + \gamma] - U[m_t(\mu^t, y^{t-1})] \geq 0 \tag{15}$$

Now following the discussion in Note 2 define

$$h(m, \bar{c}) := U(\bar{c}) - U(m) - U'(\bar{c})(\bar{c} + \gamma - m) \tag{16}$$

so that a household makes a transfer if and only if $h(m, \bar{c}) \geq 0$. It's straightforward to show (again, see Note 2) that $h(m, \bar{c})$ is strictly convex in m, strictly decreasing in m for all $m < \bar{c}$, reaching a unique minimum at $m = \bar{c}$ and strictly increasing for all $m > \bar{c}$. Moreover $h(\bar{c}, \bar{c}) < 0$ so that there are typically two solutions to $h(m, \bar{c}) = 0$ for each $\bar{c}$. Call these solutions m_L, m_H such that

$$h(m_L, \bar{c}) = 0 = h(m_H, \bar{c})$$

To summarize we have an *inaction region*, an interval (m_L, m_H) on which $h(m, \bar{c}) < 0$ such that the household makes no transfer, and *active regions* $m \leq m_L$ and $m \geq m_H$ on which $h(m, \bar{c}) \geq 0$ such that the household makes a transfer. Households with $m \geq m_H$ have an excess stock of real balances in the goods market at the beginning of the period and make a transfer *to* the asset market. Households with $m \leq m_L$ have insufficient real balances in the goods market at the beginning of the period and make a transfer *from* the asset market.

3 Equilibrium

In an equilibrium of the kind studied by Alvarez, Atkeson and Kehoe (2002) where the cash-in-advance constraint binds and no cash is held over in the asset market, the market clearing condition imply that the price level is given by $P_t(\mu^t) = \hat{M}_t(\mu^t)/Y$ so that inflation is equal to money growth. Consequently, the idiosyncratic real balances a household has in the goods market at the beginning of the period are

$$m_t(\mu^t, y^{t-1}) = \frac{y_{t-1}}{\mu_t} \tag{17}$$

and so households that make no transfer have consumption $c_t(\mu^t, y^{t-1}) = y_{t-1}/\mu_t$. Now fix a current realized money growth μ_t. For some *as-yet unknown* $\bar{c}$ we solve for cutoff points y_L, y_H using $y_L = m_L\mu_t$ and $y_H = m_H\mu_t$ (both implicitly functions of $\bar{c}$). We then solve for $\bar{c}$ using the resource constraint

$$Y = \int_0^{y_L} (\bar{c} + \gamma) f(y) dy + \frac{1}{\mu_t} \int_{y_L}^{y_H} y f(y) dy + \int_{y_H}^{\infty} (\bar{c} + \gamma) f(y) dy \tag{18}$$

This implies $\bar{c}$ is a function only of the current μ_t and an argument along the lines given in Note 2 shows that there is a unique solution $\bar{c}(\mu_t)$ for each μ_t. This solution in turn pins down the transfer indicator functions $z_t(\mu^t, y^{t-1})$ and transfers $x_t(\mu^t, y^{t-1})$ for every type of household and all the other equilibrium objects.

In particular equilibrium asset prices, as summarized by the date zero prices $Q_t(\mu^t)$ can be read off the first order conditions for consumption of active households

$$\beta^t \frac{U'[\bar{c}(\mu_t)]}{M_t(\mu^t)} g_t(\mu^t) Y = \lambda Q_t(\mu^t)$$

where λ is the single Lagrange multiplier on ex ante identical households' intertemporal budget constraint. Once we've used the resource constraint to pin down the mapping $\bar{c}(\mu_t)$ we can use this first order condition to calculate asset prices.

We generally do not have a closed form solution for $\bar{c}(\mu_t)$. Moreover, it is difficult to calculate the sign of $\bar{c}'(\mu_t)$ in general. On the one hand, higher inflation redistributes real consumption from inactive households who have consumption $c_t(\mu^t, y^{t-1}) = y_{t-1}/\mu_t$ and so with a fixed aggregate endowment this is a force that tends to increase the consumption of active households $\bar{c}(\mu_t)$. On the other hand, higher inflation also tends to increase the number of households that make a transfer by changing the cutoffs y_L, y_H and this tends to lower $\bar{c}(\mu_t)$. Alvarez, Atkeson and Kehoe (2002) assume that the former effect dominates so that an increase in money growth increases the consumption of active households.

We will discuss the asset pricing implications of this model in detail in the next note.

Chris Edmond

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