Review of Basic Cash-in-Advance Models
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This note reviews the use of a cash-in-advance constraint to introduce a role for money into an otherwise standard complete markets model. For simplicity, I will work only with exchange economies. The extension to production economies is straightforward. This exposition is a slight variant of Lucas (1982, 1984).

1 Closed economy version

Let there be countable dates, $t = 0, 1, 2...$ and let a state of nature be indexed by s_t. A history is a vector $s^t = (s_0, s_1, ..., s_t) = (s^{t-1}, s_t)$. The unconditional probability of a history s^t being realized as of date zero is denoted $f_t(s^t)$. The initial state s_0 is known as of date zero.

Agents and endowments. There is a representative household and a government. For expositional purposes, the representative household is split into a 'worker' and a 'shopper'. The worker receives a stochastic endowment $y_t(s^t)$ of a single non-storable good. The household is prohibited from directly consuming its own endowment. Instead, the worker sells the endowment for money which is used by the shopper to buy consumption. More detail on this below.

Markets. There are two markets, an asset market and a goods markets. Households trade money and nominal bonds in the asset market (i.e., bonds that promise payment in the arbitrary unit of account, say *dollars*). The government injects money into the asset market via *open market operations*. In the goods market, the household's shopper uses money to make purchases of the consumption good and the household's worker sells their endowment for money.

Money. Let $\hat{M}_t(s^t)$ denote the exogenous supply of money with growth rate

$$\mu_t(s^t) := \log\left(\frac{\hat{M}_t(s^t)}{\hat{M}_{t-1}(s^{t-1})}\right)$$

Also, let $\tau_t(s^t)$ denote lump-sum taxes. The state s_t in any period is the realization of money growth, endowments, and lump-sum taxes

$$s_t = (\mu_t, y_t, \tau_t)$$

Cash-in-advance. In order for money to be valued in equilibrium, we need to impose some kind of friction that will give rise to its use. The simplest friction involves requiring consumers to buy goods with money. These are cash-in-advance constraints

$$P_t(s^t)c_t(s^t) \leq M_t(s^t)$$

In this expression, $P_t(s^t)$ denotes the price level while $c_t(s^t)$ denotes aggregate real consumption.

Timing. Each period t, first the state s_t is realized. Then any bonds due are paid out in the asset market. Households then trade new bonds and money in the asset market and governments make their announced open market operations. Once asset market trade is finished, households 'split' into workers and shoppers. The workers sell the household's endowment for money and then bring that money back home. Simultaneously, the shoppers use the money previously acquired from asset market transactions to make goods market purchases. The households then jointly consume the shopper's purchases. The stock of money held over until the next period is the sum of unspent money by the shopper plus money brought back by the worker.

Household flow budget constraints. In the asset market, households have money held over from the previous period plus payments from bond holdings and they can use these funds to buy more bonds or domestic money or pay taxes, that is

$$\begin{aligned} & M_t(s^t) + P_t(s^t)\tau_t(s^t) + \sum_{s'} q_t(s^t, s')B_{t+1}(s^t, s') \\ \leq\ & P_{t-1}(s^{t-1})y_{t-1}(s^{t-1}) + M_{t-1}(s^{t-1}) - P_{t-1}(s^{t-1})c_{t-1}(s^{t-1}) + B_t(s^{t-1}, s_t) \end{aligned}$$

Household preferences. The representative household has preferences over streams of consumption $c := \{c_t(s^t)\}_{t=0}^{\infty}$. These preferences are given by the expected utility function

$$u(c) := \sum_{t=0}^{\infty} \sum_{s^t} \beta^t U[c_t(s^t)] f_t(s^t), \qquad 0 < \beta < 1$$

The household maximizes $u(c)$ by choice of consumption, bond and money holdings subject to its flow budget constraints and cash-in-advance constraints.

Government flow budget constraints. The government has beginning of the period liabilities $\hat{B}_t(s^{t-1}, s_t)$ which can be covered by printing money, raising taxes, or by selling more state contingent bonds

$$\hat{B}_t(s^{t-1}, s_t) \leq \hat{M}_t(s^t) - \hat{M}_{t-1}(s^{t-1}) + P_t(s^t)\tau_t(s^t) + \sum_{s'} q_t(s^t, s')\hat{B}_{t+1}(s^t, s')$$

Market clearing. The goods market clears when shoppers buy all of the worker's sales of the endowment of goods

$$c_t(s^t) \;=\; y_t(s^t)$$

The asset market clears when households buy all government sales of all nominal state contingent bonds

$$B_{t+1}(s^t, s') \;=\; \hat{B}_{t+1}(s^t, s'), \qquad \text{all } s'$$

and households hold all money

$$M_t(s^t) \;=\; \hat{M}_t(s^t)$$

Optimization. Let $\lambda_t(s^t) \geq 0$ denote the multiplier on the relevant household flow budget constraint and let $\eta_t(s^t) \geq 0$ denote the multiplier on the relevant cash-in-advance constraint. The interesting first order conditions for the household's problem include

$$c_t(s^t) \; : \qquad \beta^t U'[c_t(s^t)] f_t(s^t) = P_t(s^t)\left[\eta_t(s^t) + \sum_{s'} \lambda_{t+1}(s^t, s')\right] \tag{1}$$

$$M_t(s^t) \; : \qquad \lambda_t(s^t) = \eta_t(s^t) + \sum_{s'} \lambda_{t+1}(s^t, s') \tag{2}$$

$$B_{t+1}(s^t, s') \; : \qquad \lambda_t(s^t) q_t(s^t, s') = \lambda_{t+1}(s^t, s') \tag{3}$$

Before combining these conditions with market clearing to solve the model, it's worth spending a moment to notice some of the key implications of household optimization.

Marginal utility of a dollar. Combining first order conditions (1) and (2) gives

$$\beta^t \frac{U'[c_t(s^t)]}{P_t(s^t)} f_t(s^t) = \lambda_t(s^t)$$

This is a fundamental relationship.

Nominal pricing kernel. Combining first order conditions (1) and (3) gives the price at s^t of a claim to a dollar in state s' at date $t+1$ in terms of the multipliers on the flow budget constraints, and hence in terms of marginal utilities. This is the (one-period) nominal pricing kernel

$$q_t(s^t, s') = \frac{\lambda_{t+1}(s^t, s')}{\lambda_t(s^t)} = \beta \frac{U'[c_{t+1}(s^t, s')]}{U'[c_t(s^t)]} \frac{P_t(s^t)}{P_{t+1}(s^t, s')} \frac{f_{t+1}(s^t, s')}{f_t(s^t)}$$

The safe nominal interest rate is then

$$\begin{aligned}\frac{1}{1+i_t(s^t)} &= \sum_{s'} q_t(s^t,s') \\ &= \sum_{s'} \left\{ \beta \frac{U'[c_{t+1}(s^t,s')]}{U'[c_t(s^t)]} \frac{P_t(s^t)}{P_{t+1}(s^t,s')} \frac{f_{t+1}(s^t,s')}{f_t(s^t)} \right\}\end{aligned}$$

or more informally

$$\frac{1}{1+i_t} = \mathbb{E}_t \left\{ \beta \frac{U'(c_{t+1})}{U'(c_t)} \frac{P_t}{P_{t+1}} \right\}$$

The random variable inside the conditional expectation is the nominal stochastic discount factor (SDF) for this model. Prices of other nominal bonds of longer maturity can then be constructed by iterating appropriately on this one-period SDF. More details on this in later notes.

Real pricing kernel. We use the price level $P_t(s^t)$ to convert claims to a dollar into claims to real consumption so that

$$q_t(s^t,s') \frac{P_{t+1}(s^t,s')}{P_t(s^t)}$$

is the price at s^t of a claim to a unit of real consumption in state s' at date $t+1$, the real pricing kernel. Given this, the real risk free rate is

$$\begin{aligned}\frac{1}{1+r_t(s^t)} &= \sum_{s'} q_t(s^t,s') \frac{P_{t+1}(s^t,s')}{P_t(s^t)} \\ &= \sum_{s'} \left\{ \beta \frac{U'[c_{t+1}(s^t,s')]}{U'[c_t(s^t)]} \frac{f_{t+1}(s^t,s')}{f_t(s^t)} \right\}\end{aligned}$$

or more informally

$$\frac{1}{1+r_t} = \mathbb{E}_t \left\{ \beta \frac{U'(c_{t+1})}{U'(c_t)} \right\}$$

The random variable inside the conditional expectation is the real SDF for this model.

Fisher equations and inflation risk premia. In undergraduate discussion of interest rates, we say that the real interest rate r_t is the nominal interest rate i_t less expected inflation, say $\mathbb{E}_t\{\pi_{t+1}\}$ where inflation $\pi_{t+1} := \log(P_{t+1}/P_t)$ is the rate of growth of the price level. If so, $i_t = r_t + \mathbb{E}_t\{\pi_{t+1}\}$. Do we get this *Fisher equation* in the basic cash-in-advance model? Not quite.

The relationship between real and nominal interest rates is

$$\frac{1}{1+i_t} = \mathbb{E}_t \left\{ \beta \frac{U'(c_{t+1})}{U'(c_t)} \right\} \mathbb{E}_t \left\{ \frac{P_t}{P_{t+1}} \right\} + \mathrm{Cov}_t \left\{ \beta \frac{U'(c_{t+1})}{U'(c_t)}, \frac{P_t}{P_{t+1}} \right\}$$

or

$$\frac{1}{1+i_t} = \frac{1}{1+r_t} \mathbb{E}_t \left\{ \frac{P_t}{P_{t+1}} \right\} + \mathrm{Cov}_t \left\{ \beta \frac{U'(c_{t+1})}{U'(c_t)}, \frac{P_t}{P_{t+1}} \right\}$$

Thus we *do not get an exact Fisher equation* that mechanically links real interest rates to nominal interest rates and expected inflation. We also have to account for the covariance between inflation and the real SDF.

The covariance between inflation and the real SDF

$$\mathrm{Cov}_t\left\{\beta\frac{U'(c_{t+1})}{U'(c_t)},\frac{P_t}{P_{t+1}}\right\}$$

is an *inflation risk premium*. In log-linearized (or perfect foresight) versions of cash-in-advance models, this covariance term will be zero. Specifically, in a perfect foresight version the relationship between interest rates and inflation is just

$$\frac{1}{1+i_t}=\frac{1}{1+r_t}\frac{P_t}{P_{t+1}}$$

or

$$\log(1+i_t)=\log(1+r_t)+\pi_{t+1}$$

Then using the first order approximation $\log(1+x)=x$ for $x\approx 0$, we have

$$i_t=r_t+\pi_{t+1}$$

where π_{t+1} should be interpreted as expected inflation. This is the undergraduate version of the Fisher equation. Note that in a log-linearized stochastic model we would have instead $i_t = r_t+\mathbb{E}_t\{\pi_{t+1}\}$. To get this last expression we are asserting that $\log\mathbb{E}_t\{P_t/P_{t+1}\}\approx\mathbb{E}_t\{\log(P_t/P_{t+1})\}$, i.e, we are asserting that the variance of inflation is negligible.

Money demand and velocity. A money demand schedule is an equilibrium relationship between (i) *real money*, defined as $m := M/P$, (ii) a measure of the opportunity cost of holding money, and (iii) a measure of real activity. A typical relationship would be something like $m = L(i, c)$ where the nominal interest rate measures the opportunity cost of money and real consumption measures real activity. Note: all of these variables are endogenous.

The cash-in-advance model has very straightforward predictions for money demand. If the cash-in-advance constraint is binding, then households have

$$P_t(s^t)c_t(s^t)=M_t(s^t)$$

or real balances

$$m_t(s^t):=\frac{M_t(s^t)}{P_t(s^t)}=c_t(s^t)$$

so for this model the function $L(i, c)$ just equals c for all i. Unlike some specifications, the demand for real balances is generally interest inelastic (so long as the cash-in-advance constraint is binding).

Recall Fisher's *exchange equation*, $Mv = Pc$, which defines the velocity of money per period, again using consumption as a measure of real transactions. Then this model implies $v_t(s^t) = 1$ for every date and state if the cash-in-advance constraint binds.

When does the cash-in-advance constraint bind? Clearly, we must have $\eta_t(s^t) > 0$. But according to the first order condition (2), this is just the same as

$$\eta_t(s^t) = \lambda_t(s^t) - \sum_{s'} \lambda_{t+1}(s^t, s') > 0$$

So long as the marginal utility of a dollar is positive, $\lambda_t(s^t) > 0$, we can divide both sides by $\lambda_t(s^t)$ to get

$$\eta_t(s^t) > 0 \Longleftrightarrow \frac{\eta_t(s^t)}{\lambda_t(s^t)} = 1 - \sum_{s'} \frac{\lambda_{t+1}(s^t, s')}{\lambda_t(s^t)} > 0$$

But

$$1 - \sum_{s'} \frac{\lambda_{t+1}(s^t, s')}{\lambda_t(s^t)} = 1 - \frac{1}{1 + i_t(s^t)} = \frac{i_t(s^t)}{1 + i_t(s^t)}$$

So the cash-in-advance constraint is binding if and only if the nominal interest rate i is positive. Put differently, if the nominal interest rate is positive, money (a safe nominal asset) is *dominated in rate of return* by bonds (also a safe nominal asset) because the bonds pay interest while money does not. Optimizing households will never have any money unspent in the goods market, never have $Pc < M$ if the nominal interest rate is positive, because they could better use that unspent money to buy bonds. This is the reason the quantity $i/(1+i)$ (or just i itself) is often said to be the *opportunity cost of money.*

Since short term nominal interest rates are generally positive in the data, this model implies that the cash-in-advance constraint should essentially always be binding and so velocity should be constant and real balances insensitive to fluctuations in nominal interest rates. In the data, we see quite close movements between velocity and nominal interest rates, which is one (amongst many!) reasons why this model is not an empirical success.

Equilibrium. Anyway, back to the model. Equilibrium allocations are trivial. Goods market clearing implies $c_t = y_t$. This gives the usual consumption-based theory of real asset prices: the real SDF is $\beta U'(y_{t+1})/U'(y_t)$ and so the real risk free rate is

$$r_t = \left\{ \mathbb{E}_t \left[\beta \frac{U'(y_{t+1})}{U'(y_t)} \right] \right\}^{-1} - 1$$

If utility is CRRA so that $U'(y) = y^{-\sigma}$ for $\sigma > 0$ then along a balanced growth path where $g_t := \log(y_{t+1}/y_t) \to \overline{g}$ all t the real risk free rate settles down to $\overline{r} \approx -\log(\beta) + \sigma \overline{g}$ as usual. Call this $\overline{r}$ the *long-run* real risk free rate.

To solve for the associated nominal asset prices we first suppose that the nominal interest rate is always positive (we'll see how to check this below). Then the cash-in-advance constraint binds so that real balances are $m_t = y_t$ too. Using asset market clearing, the price level is just

$$P_t = \frac{\hat{M}_t}{y_t}$$

and so inflation, the growth rate of the price level, is $\pi_t = \mu_t - g_t$. Given the solution for the price level, the nominal SDF is

$$\beta \frac{U'(y_{t+1})}{U'(y_t)} \frac{P_t}{P_{t+1}} = \beta \frac{U'(y_{t+1})}{U'(y_t)} \frac{y_{t+1}}{y_t} \frac{\hat{M}_t}{\hat{M}_{t+1}}$$

and the nominal risk free rate is

$$i_t = \left\{ \mathbb{E}_t \left[\beta \frac{U'(y_{t+1})}{U'(y_t)} \frac{y_{t+1}}{y_t} \frac{\hat{M}_t}{\hat{M}_{t+1}} \right] \right\}^{-1} - 1$$

Again, if utility is CRRA then along a balanced growth path where $g_t := \log(y_{t+1}/y_t) \to \overline{g}$ and $\mu_t := \log(\hat{M}_{t+1}/\hat{M}_t) \to \overline{\mu}$ all t, the nominal risk free rate settles down to $\overline{\imath} \approx -\log(\beta) + (\sigma - 1)\overline{g} + \overline{\mu}$. Since along such a balanced growth path inflation is $\overline{\pi} = \overline{\mu} - \overline{g}$, this long-run nominal risk free rate can also be written $\overline{\imath} \approx \overline{r} + \overline{\pi}$, which is a version of the usual undergraduate Fisher equation.

Real/nominal dichotomy and monetary neutrality. This model exhibits a strong dichotomy between real and nominal variables. Real consumption is equal to the endowment of goods y_t. Real balances are equal to real consumption and hence also equal y_t. Real interest rates r_t are determined by the real SDF which is a simple function of time preference and real consumption. Etc. Nominal variables do not influence real variables *at all* (though nominal variables do depend on real ones). The price level P_t is pinned down by the level of the exogenous money supply $\hat{M}_t$ and the level of the endowment of goods y_t, that is $P_t = \hat{M}_t/y_t$. Inflation is therefore the rate of change of the money supply less real consumption growth, $\pi_t = \mu_t - g_t$. Nominal interest rates are real rates plus compensation for inflation (i.e., money growth). In short, this is a model where money is *neutral* in the sense that changes in the money supply have no real effects at any horizon.

Checking that the CIA constraint binds. A pedantic detail. Given specified stochastic processes for the exogenous variables y_t and $\hat{M}_t$ we can then calculate i_t using the formula above and check that it is always positive so that the cash-in-advance constraint is binding as we assumed.

Many popular specifications fail to guarantee that the nominal interest rate is in fact always positive. To see this, suppose that we have a constant endowment $g_t = \overline{g} = 1$ all t and that money growth follows an AR1 with Gaussian shocks

$$\mu_{t+1} = (1 - \phi)\overline{\mu} + \phi\mu_t + \varepsilon_{t+1}, \qquad \varepsilon_{t+1} \sim \text{ IID and } N(0, \sigma_\varepsilon^2), \qquad 0 \leq \phi < 1$$

Then the nominal SDF is

$$\beta \mathbb{E}_t \left\{ \exp(-(1-\phi)\overline{\mu} - \phi\mu_t - \varepsilon_{t+1}) \right\} = \beta \exp(-(1-\phi)\overline{\mu} - \phi\mu_t) \exp(\sigma_\varepsilon^2/2)$$

and clearly we can find μ_t such that the cash-in-advance constraint is slack. In applied work, many people seem to ignore this and related issues.

Fisher effects and liquidity effects. Back to the big picture (what follows is taken from Edmond and Weill 2005). The term *liquidity effect* as used in macroeconomics refers to a fall in nominal interest rates following an exogenous persistent increase in narrow measures of the money supply, here $\hat{M}_t$. This is in contrast to the *Fisher effect* whereby an exogenous, persistent increase in the money supply is predicted to increase expected inflation and hence nominal interest rates.

Friedman (1968) argues that, in practice, both the liquidity effect and Fisher effect operate: a persistent increase in the money supply both reduces nominal interest rates and increases expected inflation so that the real rate also falls. Friedman speculates that following a monetary shock nominal and real rates may fall below their typical levels for up to a year, but, over time, rates will then tend to increase before tending to the levels consistent with the inflation generated by the original monetary shock. Put simply, money is argued to be neutral in the long run but not neutral in the short run.

Applied macroeconomists have interpreted Friedman (1968) as follows. In the long run real interest rates are determined by 'fundamentals' including the rate at which households discount β and average productivity growth $\overline{g}$. Consequently we should expect that long-horizon real interest rates are relatively stable and are unaffected by transitory monetary disturbances. Long-horizon nominal interest rates are this stable real rate plus expected inflation. At short horizons, however, Friedman's (1968) argument suggests that real and nominal interest rates are both volatile and positively correlated. His argument also suggests that short-horizon real rates and expected inflation are negatively correlated. Barr and Campbell (1997) provide evidence consistent with this interpretation and Cochrane (1989) provides specific evidence for liquidity effects at short horizons. More on this in class.

Partial and general equilibrium. Perhaps the easiest way to interpret Friedman (1968) is in terms of the following scenario. Suppose the monetary authority increases the money supply by conducting an unexpected outright purchase of bonds. At short horizons, nominal interest rates fall so that households are willing to hold a smaller quantity of bonds and a larger quantity of money. But this is only a partial equilibrium effect. As households spend their increased money holdings on goods, the price level increases and so real balances do not rise as fast as nominal balances. This general equilibrium effect mitigates the need for the nominal interest rate to fall. In many simple monetary models, households tend to spend money so 'fast' that the general equilibrium price level effect can completely overturn the partial equilibrium effect.

This general equilibrium price level effect is particularly strong in the basic cash-in-advance model we've been discussing. Suppose $g_t = 0$ all t (so the endowment is a constant) and money growth μ_t is IID over time with mean $\overline{\mu}$. If so, households immediately spend an unexpected increase in money on a fixed quantity of real goods. This increases the price level one-for-one with the increase in the money supply so that real balances are unchanged, $m_t = \overline{m}$ all t. In addition, because money growth is IID, expected inflation is constant $\mathbb{E}_t\{\pi_{t+1}\} = \overline{\pi} = \overline{\mu}$ all t. Taken together, constant real balances and constant expected inflation imply that the money market clears at a constant nominal interest rate, $i_t = \overline{\imath}$ all t. In this case, the general equilibrium price level effect completely offsets the partial equilibrium liquidity effect. If instead monetary growth shocks are persistent (an AR1, say) then a positive shock increases expected inflation and nominal interest rates increase. In short: there is a Fisher effect but no liquidity effect.

2 Two-country version

Unless otherwise discussed, the model is the same as above. Except:

Agents and endowments. Now there are two countries. In each country there is a representative household and a government. In each country, the representative household is again comprised of a worker and a shopper. Households in the home country use *dollars* to purchase consumption goods, while households in the foreign country use *euros* to purchase consumption goods. Home households have a stochastic endowment $y_t(s^t)$ of a single non-storable good while foreign households have an endowment $y_t^*(s^t)$ of *the same* non-storable good.

Markets. There are two markets, an international asset market and an international goods market. Households trade the two monies and dollar and euro denominated bonds in the asset market (i.e., bonds that promise payment in the respective currency). The two governments inject money into the asset market via open market operations. In the goods market, households use their local currency to make purchases of goods.

Monies. Let $\hat{M}_t(s^t)$ denote the exogenous supply of dollars in the home country with growth rate $\mu_t(s^t)$ and let $M_t^*(s^t)$ denote the corresponding exogenous supply of euros in the foreign country with growth rate $\mu_t^*(s^t)$. Also, let $\tau_t(s^t)$ and $\tau_t^*(s^t)$ denote lump-sum taxes in each country. The state s_t in any period is the realization of money growth, endowments, and lump-sum taxes

$$s_t = (\mu_t, \mu_t^*, y_t, y_t^*, \tau_t, \tau_t^*)$$

Cash-in-advance. For the representative consumer in the home country, we have, as before,

$$P_t(s^t)c_t(s^t) \leq M_t(s^t)$$

Similarly for the foreign country

$$P_t^*(s^t)c_t^*(s^t) \leq M_t^*(s^t)$$

where $P_t^*(s^t)$ denotes the price level in euros.

Household flow budget constraints. In the asset market, households have money held over from the previous period plus payments from bond holdings and they can use these funds to buy more dollar or euro bonds or domestic money or pay taxes, that is

$$\begin{aligned} & M_t(s^t) + P_t(s^t)\tau_t(s^t) + \sum_{s'} q_t(s^t, s')B_{t+1}^{\mathrm{H}}(s^t, s') + \mathcal{E}_t(s^t)\sum_{s'} q_t^*(s^t, s')B_{t+1}^{\mathrm{H}*}(s^t, s') \\ \leq\ & P_{t-1}(s^{t-1})y_{t-1}(s^{t-1}) + M_{t-1}(s^{t-1}) - P_{t-1}(s^{t-1})c_{t-1}(s^{t-1}) + B_t^{\mathrm{H}}(s^{t-1}, s_t) + \mathcal{E}_t(s^t)B_t^{\mathrm{H}*}(s^{t-1}, s_t) \end{aligned}$$

Here $\mathcal{E}_t(z^t)$ is the nominal exchange rate between euros and dollars. There is an analogous constraint for the foreign country, see below.

Government budget constraints. The home government has beginning of the period liabilities $\hat{B}_t(z^{t-1}, z_t)$ which can be covered by printing money, raising taxes, or by selling more state contingent bonds

$$\hat{B}_t(s^{t-1}, s_t) \leq \hat{M}_t(s^t) - \hat{M}_{t-1}(s^{t-1}) + P_t(s^t)\tau_t(s^t) + \sum_{s'} q_t(s^t, s')\hat{B}_{t+1}(s^t, s')$$

Similarly, the foreign government faces

$$\hat{B}_t^*(s^{t-1}, s_t) \leq \hat{M}_t^*(s^t) - \hat{M}_{t-1}^*(s^{t-1}) + P_t^*(s^t)\tau_t^*(s^t) + \sum_{s'} q_t^*(s^t, s')\hat{B}_{t+1}^*(s^t, s')$$

Market clearing. International goods market clearing means

$$c_t(s^t) + c_t^*(s^t) = y_t(s^t) + y_t^*(s^t)$$

The international asset market clears when home and foreign households buy government sales of all nominal state contingent bonds

$$\begin{aligned} B_{t+1}^{\mathrm{H}}(s^t, s') + B_{t+1}^{\mathrm{F}}(s^t, s') &= \hat{B}_{t+1}(s^t, s'), \qquad \text{all } s' \\ B_{t+1}^{\mathrm{H}*}(s^t, s') + B_{t+1}^{\mathrm{F}*}(s^t, s') &= \hat{B}_{t+1}^*(s^t, s'), \qquad \text{all } s' \end{aligned}$$

and households hold all money

$$\begin{aligned} M_t(s^t) &= \hat{M}_t(s^t) \\ M_t^*(s^t) &= \hat{M}_t^*(s^t) \end{aligned}$$

Here B_{t+1}^{H} denotes home demand for dollar bonds, B_{t+1}^{F} denotes foreign demand for dollar bonds, $B_{t+1}^{\mathrm{H}*}$ denotes home demand for euro bonds, and $B_{t+1}^{\mathrm{F}*}$ denotes foreign demand for euro bonds. The aggregate supplies $\hat{B}_{t+1}$ and $\hat{B}_{t+1}^*$ are the bond sales of the respective governments.

Optimization. Let $\lambda_t(s^t) \geq 0$ denote the multiplier on the home country budget constraint and let $\eta_t(s^t) \geq 0$ denote the multiplier on the cash-in-advance constraint. The interesting first order conditions for the home country problem include

$$c_t(s^t) \quad : \quad \beta^t U'[c_t(s^t)] f_t(s^t) = P_t(s^t) \left[\eta_t(s^t) + \sum_{s'} \lambda_{t+1}(s^t, s') \right] \tag{4}$$

$$M_t(s^t) \quad : \quad \lambda_t(s^t) = \eta_t(s^t) + \sum_{s'} \lambda_{t+1}(s^t, s') \tag{5}$$

$$B_{t+1}^{\mathrm{H}}(s^t, s') \quad : \quad \lambda_t(s^t) q_t(s^t, s') = \lambda_{t+1}(s^t, s') \tag{6}$$

$$B_{t+1}^{\mathrm{H}*}(s^t, s') \quad : \quad \lambda_t(s^t) \mathcal{E}_t(s^t) q_t^*(s^t, s') = \lambda_{t+1}(s^t, s') \mathcal{E}_{t+1}(s^t, s') \tag{7}$$

Before turning to analogous conditions for the foreign country, we first discuss some important implications that we get from the home country alone.

Dollar and euro pricing kernels. The relationship between the pricing kernel $q_t(s^t, s')$ for dollar denominated assets and the pricing kernel $q_t^*(s^t, s')$ for euro denominated assets is

$$q_t(s^t, s') = q_t^*(s^t, s') \frac{\mathcal{E}_t(s^t)}{\mathcal{E}_{t+1}(s^t, s')}$$

To convert an asset price in euros into an asset price in dollars, we have to take into account the nominal exchange rate movement between periods. As an example, consider the following calculation: suppose that I want to ensure that I have a dollar tomorrow in state s'. I could just buy a bond that pays a dollar in that state, such a bond has price $q_t(s^t, s')$ today. But another way to get a dollar in state s' is to buy just the right number of euro bonds so that when I convert euros to dollars in state s' I get exactly one dollar. A dollar tomorrow will require $\frac{1}{\mathcal{E}_{t+1}(s^t, s')}$ euros at $t+1$ which can be bought for $\frac{q_t^*(s^t, s')}{\mathcal{E}_{t+1}(s^t, s')}$ euros at t. But $\frac{q_t^*(s^t, s')}{\mathcal{E}_{t+1}(s^t, s')}$ euros at t is equal to $q_t^*(s^t, s') \frac{\mathcal{E}_t(s^t)}{\mathcal{E}_{t+1}(s^t, s')}$ dollars at t. So I could lay out this many dollars to make sure that I have a dollar in s' at $t+1$. If there are to be no arbitrage profits, it had better be the case that this is equal to the original dollar price $q_t(s^t, s')$.

The safe nominal interest rate on *euro bonds* is given by

$$\begin{aligned}\frac{1}{1+i_t^*(s^t)} &= \sum_{s'} q_t^*(s^t,s') \\ &= \sum_{s'} \left\{ q_t(s^t,s')\frac{\mathcal{E}_{t+1}(s^t,s')}{\mathcal{E}_t(s^t)} \right\} \\ &= \sum_{s'} \left\{ \beta\frac{U'[c_{t+1}(s^t,s')]}{U'[c_t(s^t)]}\frac{P_t(s^t)}{P_{t+1}(s^t,s')}\frac{\mathcal{E}_{t+1}(s^t,s')}{\mathcal{E}_t(s^t)}\frac{f_{t+1}(s^t,s')}{f_t(s^t)} \right\}\end{aligned}$$

or more informally

$$\frac{1}{1+i_t^*} = \mathbb{E}_t\left\{ \beta\frac{U'(c_{t+1})}{U'(c_t)}\frac{P_t}{P_{t+1}}\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t} \right\}$$

Notice that this is a relationship between foreign nominal interest rates and the home country nominal pricing kernel.

Forward exchange rates and covered interest parity. The (one-period) *forward exchange rate* is an agreement to purchase one euro at $t+1$ with a number $\mathcal{F}_t(s^t)$ of dollars at time t. That is, the forward rare is a one-period-ahead contract to lock in the spot rate at which you will trade next period. Given a complete set of state contingent nominal securities, this asset is redundant and we can figure out how to price it given the spot nominal exchange rate and the nominal interest rates in each country. An agent can borrow one dollar, use it to buy $\frac{1}{\mathcal{E}_t(s^t)}$ euros, use those euros to buy bonds that pay $1+i_t^*(s^t)$ each for a total of $\frac{1+i_t^*(s^t)}{\mathcal{E}_t(s^t)}$ at $t+1$. If the forward rate is $\mathcal{F}_t(s^t)$, then this total can be turned into a return of $[1+i_t^*(s^t)]\frac{\mathcal{F}_t(s^t)}{\mathcal{E}_t(s^t)}$ dollars *for sure* at date $t+1$. But we already know that the price of a dollar for sure at date $t+1$ is $\frac{1}{1+i_t(s^t)}$. If there are to be no arbitrage profits, it had better be the case that these two prices are the same

$$\frac{1}{1+i_t(s^t)} = \frac{1}{1+i_t^*(s^t)}\frac{\mathcal{E}_t(s^t)}{\mathcal{F}_t(s^t)}$$

The term on the left is the price of a bond that pays a dollar for sure at $t+1$, the term on the right is the price of a contract that delivers a dollar for sure via the appropriate euro assets with the spot exchange rate at which the payment is made at $t+1$ locked in forward. Thus, both contracts are riskless. This is often written

$$\frac{\mathcal{F}_t(s^t)}{\mathcal{E}_t(s^t)} = \frac{1+i_t(s^t)}{1+i_t^*(s^t)}$$

and is the so-called *covered interest parity* (CIP) condition. Given spot rates and the two nominal interest rates, we can back out the forward rate that is consistent with no arbitrage profits. Empirically, a relationship of this form holds very well (transaction costs in the relevant asset markets are very very small). Using $\log(1+x) = x$ for $x \approx 0$ we have that approximately

$$f_t - e_t = i_t - i_t^*$$

where

$$\begin{aligned} f_t &:= \log(\mathcal{F}_t) \\ e_t &:= \log(\mathcal{E}_t) \end{aligned}$$

are the log forward and log spot nominal exchange rates. The one period log forward/spot differential is just the interest differential on riskless bonds of one period maturity.

Uncovered interest parity. Undergraduate textbooks introduce the concept of *uncovered interest parity* (UIP), which states that the expected depreciation between two currencies is determined by the nominal interest differential

$$i_t - i_t^* = \mathbb{E}_t\{\Delta e_{t+1}\}, \qquad \Delta e_{t+1} := e_{t+1} - e_t$$

As with the Fisher equation in the closed economy model, we do not quite get the undergraduate UIP result unless we make some additional assumptions. To see this, recall that the safe dollar bond price is

$$\frac{1}{1+i_t} = \mathbb{E}_t\left\{\beta\frac{U'(c_{t+1})}{U'(c_t)}\frac{P_t}{P_{t+1}}\right\}$$

while the safe euro bond price is

$$\frac{1}{1+i_t^*} = \mathbb{E}_t\left\{\beta\frac{U'(c_{t+1})}{U'(c_t)}\frac{P_t}{P_{t+1}}\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}\right\}$$

Expanding the conditional expectation on the right

$$\begin{aligned} \frac{1}{1+i_t^*} &= \mathbb{E}_t\left\{\beta\frac{U'(c_{t+1})}{U'(c_t)}\frac{P_t}{P_{t+1}}\right\}\mathbb{E}_t\left\{\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}\right\} + \mathrm{Cov}_t\left\{\beta\frac{U'(c_{t+1})}{U'(c_t)}\frac{P_t}{P_{t+1}}, \frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}\right\} \\ &= \frac{1}{1+i_t}\mathbb{E}_t\left\{\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}\right\} + \mathrm{Cov}_t\left\{\beta\frac{U'(c_{t+1})}{U'(c_t)}\frac{P_t}{P_{t+1}}, \frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}\right\} \end{aligned}$$

We do not get the textbook uncovered interest parity relationship. Instead, only if the covariance term is zero and exchange rate changes have small variance do we get

$$i_t - i_t^* = \mathbb{E}_t\{\Delta e_{t+1}\}$$

This hypothesis says that interest rate differentials merely reflect expected exchange rate movements. For example, if the nominal interest rate in the home country is high, that merely reflects the *expected depreciation* of the dollar against the euro. If the covered and uncovered interest parity conditions both held, we ought to get the relationship

$$f_t - e_t = \mathbb{E}_t\left\{\Delta e_{t+1}\right\}$$

or

$$f_t = \mathbb{E}_t\{e_{t+1}\}$$

That is, if both interest parity conditions held, forward rates should be approximately equal to expected spot rates.

Forward premium anomaly and exchange rate risk premia. But this relationship certainly does not hold in the data. One of the major puzzles in international macroeconomics is the so-called *forward premium anomaly.* Countries with relatively high interest rates seem to experience nominal exchange rate *appreciations*, whereas the covered interest parity condition tells us that relatively high nominal interest rates, $i_t - i_t^* > 0$, should go hand in hand with $\mathbb{E}_t\{\Delta e_{t+1}\} > 0$, that is, with expected nominal exchange rate depreciations. Regressions of the form

$$e_{t+1} - e_t = \alpha_0 + \alpha_1(f_t - e_t) + \text{noise}$$

typically estimate $\alpha_1 = -0.90$ or thereabouts, not the $\alpha_1 = 1$ that we expect from our theory. (Not even the sign is 'right'!). The term

$$f_t - \mathbb{E}_t\{e_{t+1}\}$$

is often said to be an *exchange rate risk premium.* A huge literature, following Fama (1984), studies the properties of this premium. Much more on this later in the course.

Now let's turn back to optimization be foreign households. They have the same utility function as home households, namely

$$u(c^*) := \sum_{t=0}^{\infty}\sum_{s^t}\beta^t U[c_t^*(s^t)]f_t(s^t), \qquad 0 < \beta < 1$$

Their sequence of flow budget constraints are

$$\begin{aligned} & \mathcal{E}_t(s^t)\left[M_t^*(s^t) + P_t^*(s^t)\tau_t^*(s^t)\right] + \sum_{s'} q_t(s^t,s')B_{t+1}^{\mathrm{F}}(s^t,s') + \mathcal{E}_t(s^t)\sum_{s'} q_t^*(s^t,s')B_{t+1}^{*\mathrm{F}}(s^t,s') \\ \leq\ & \mathcal{E}_t(s^t)[P_{t-1}^*(s^{t-1})y_{t-1}^*(s^{t-1}) + M_{t-1}^*(s^{t-1}) - P_{t-1}^*(s^{t-1})c_{t-1}^*(s^{t-1})] + B_t^{\mathrm{F}}(s^{t-1},s_t) + \mathcal{E}_t(s^t)B_t^{*\mathrm{F}}(s^{t-1},s_t) \end{aligned}$$

where for comparison with the home country I have expressed everything in dollars.

Then we have the first order conditions

$$c_t^*(s^t) \quad : \quad \beta^t U'[c_t^*(s^t)]f_t(s^t) = \mathcal{E}_t(s^t)P_t^*(s^t)\left[\eta_t^*(s^t) + \sum_{s'}\lambda_{t+1}^*(s^t,s')\right] \tag{8}$$

$$M_t^*(s^t) \quad : \quad \lambda_t^*(s^t) = \eta_t^*(s^t) + \sum_{s'}\lambda_{t+1}^*(s^t,s') \tag{9}$$

$$B_{t+1}^{\mathrm{F}}(s^t,s') \quad : \quad \lambda_t^*(s^t)q_t(s^t,s') = \lambda_{t+1}^*(s^t,s') \tag{10}$$

$$B_{t+1}^{\mathrm{F}*}(s^t,s') \quad : \quad \lambda_t^*(s^t)\mathcal{E}_t(s^t)q_t^*(s^t,s') = \lambda_{t+1}^*(s^t,s')\mathcal{E}_{t+1}(s^t,s') \tag{11}$$

Compare these to their equivalents from the home country.

In particular, the marginal utility of a euro is given by

$$\beta^t\frac{U'[c_t^*(s^t)]}{P_t^*(s^t)}f_t(s^t) = \mathcal{E}_t(s^t)\lambda_t^*(s^t)$$

Real exchange rate. The real exchange rate is defined by

$$\mathcal{Q}_t\left(s^t\right) := \mathcal{E}_t(s^t)\frac{P_t^*(s^t)}{P_t(s^t)}$$

Using the marginal utility of a dollar for the the home country and the marginal utility of a euro for the foreign country we have

$$\mathcal{Q}_t\left(s^t\right) := \mathcal{E}_t(s^t)\frac{P_t^*(s^t)}{P_t(s^t)} = \frac{U'[c_t^*(s^t)]}{U'[c_t(s^t)]}\frac{\lambda_t(s^t)}{\lambda_t^*(s^t)}$$

Since the growth rate of the Lagrange multipliers is determined by the pricing kernels we also have

$$q_t(s^t, s') = \frac{\lambda_{t+1}^*(s^t, s')}{\lambda_t^*(s^t)} = \frac{\lambda_{t+1}(s^t, s')}{\lambda_t(s^t)}$$

So the *real depreciation* in the exchange rate can be expressed as

$$\frac{\mathcal{Q}_{t+1}\left(s^t, s'\right)}{\mathcal{Q}_t\left(s^t\right)} = \frac{U'[c_{t+1}^*(s^t, s')]}{U'[c_t^*(s^t)]}\frac{U'[c_t(s^t)]}{U'[c_{t+1}(s^t, s')]}$$

With complete asset markets, as here, changes in the real exchange rate are equal to changes in the ratios of marginal utility across foreign and home consumers. If utility is CRRA so that $U'(c) = c^{-\sigma}$ for $\sigma > 0$ then

$$\Delta q_{t+1} = \sigma(\Delta \log c_{t+1} - \Delta \log c_{t+1}^*)$$

where $q_t := \log(\mathcal{Q}_t)$. Real exchange rate growth volatility should be explained by differentials in consumption growth volatility. In the data, for reasonable assumptions about relative risk aversion, real exchange rates are much too volatile to be accounted for by the low observed volatility in national consumption growth rates. Yet another reason why the model is not an empirical success.

Equilibrium in the two-country model (sketch). I say 'sketch' because I take as given some basic results about risk sharing in complete markets economies and don't re-derive them in this context. To simplify calculations, specialize to CRRA utility with $U'(c) = c^{-\sigma}$ for $\sigma > 0$. Denote the *aggregate* (global) endowment by

$$x_t(s^t) := y_t(s^t) + y_t^*(s^t)$$

To solve the model, note that with complete markets we will have consumption allocations that are time-invariant functions of the aggregate endowment $x_t(s^t)$, written

$$c_t(s^t) = \omega x_t(s^t)$$

and

$$c_t^*(s^t) = \omega^* x_t(s^t)$$

where $\omega > 0$ and $\omega^* > 0$ are constants that depend on the present value of each country's initial (date zero) wealth and risk aversion σ. Note $\omega + \omega^* = 1$ so that $c_t(s^t) + c_t^*(s^t) = x_t(s^t)$. These equilibrium allocations imply consumption growth for each country is equal, $\Delta \log c_t = \Delta \log c_t^* = \Delta \log x_t$. Therefore the real exchange rate is a constant

$$\Delta q_t = \sigma(\Delta \log c_{t+1} - \Delta \log c_{t+1}^*) = \sigma(\Delta \log x_t - \Delta \log x_t) = 0$$

To determine the *level* of the real exchange rate, note that with complete markets we can write the multipliers on the flow budget constraints in the form

$$\lambda_t(s^t) = \lambda_0 \beta^t f_t(s^t), \qquad \text{and} \qquad \lambda_t^*(s^t) = \lambda_0^* \beta^t f_t(s^t)$$

for some constants λ_0, λ_0^*. Given this we can use the expressions for the marginal utility of a dollar and the marginal utility of a euro to write

$$P_t \lambda_0 = U'(c_t) = (\omega x_t)^{-\sigma}$$

and

$$\mathcal{E}_t P_t^* \lambda_0^* = U'(c_t^*) = (\omega^* x_t)^{-\sigma}$$

Taking the ratios of these two expressions we have

$$\frac{P_t}{\mathcal{E}_t P_t^*} \frac{\lambda_0}{\lambda_0^*} = \left(\frac{\omega}{\omega^*}\right)^{-\sigma}$$

or

$$\mathcal{Q} = \mathcal{E}_t \frac{P_t^*}{P_t} = \left(\frac{\omega}{\omega^*}\right)^{\sigma} \frac{\lambda_0}{\lambda_0^*}$$

Now let's solve for the nominal variables. If the nominal interest rate is positive in each country, both cash-in-advance constraints bind and so the price levels are

$$P_t = \frac{\hat{M}_t}{c_t} = \frac{1}{\omega} \frac{\hat{M}_t}{x_t}$$

and

$$P_t^* = \frac{\hat{M}_t^*}{c_t^*} = \frac{1}{\omega^*} \frac{\hat{M}_t^*}{x_t}$$

With the solutions for the price levels and real consumption we can compute the real and nominal SDF and solve for the nominal interest rates i_t and i_t^* just as in the closed economy model above. The calculations imply that the real interest rate is identical in each country (why?)

The price level solutions imply that domestic inflation is given by domestic money growth in excess of aggregate endowment growth and similarly foreign inflation is given by foreign money growth in excess of aggregate endowment growth. Therefore, the inflation differential between domestic and foreign is

$$\pi_t - \pi_t^* = \mu_t - \mu_t^*$$

Moreover, since the real exchange rate is constant at $\mathcal{Q}$ we can write

$$\mathcal{E}_t = \mathcal{Q}\frac{P_t}{P_t^*} = \mathcal{Q}\frac{\omega^*}{\omega}\frac{\hat{M}_t}{\hat{M}_t^*}$$

The level of the nominal exchange rate is a constant times the ratio of the two money supplies and so the depreciation of the nominal exchange rate is given by the inflation/money growth differential

$$\Delta e_t = \mu_t - \mu_t^* = \pi_t - \pi_t^*$$

The two-country cash-in-advance economy also exhibits a strong dichotomy between real and nominal variables. In particular, here we have the additional implication that the nominal exchange rate fully adjusts to accommodate changes in the relative money supply. The real exchange rate is constant, totally unaffected by fluctuations in the relative money supplies.

Chris Edmond

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