Monetary Economics: Problem Set #5
Solutions

This problem set is marked out of 100 points. The weight given to each part is indicated below. Please contact me asap if you have any questions.

1. **Structured finance**. Suppose there are two bonds and that each pays \$1 cash or not. The probability of getting \$1 is 0.95 and is independent across bonds.

   (a) Explain how a financial intermediary can sell prioritised junior j and senior s claims to \$1 against the possible cash flows from a portfolio of these two bonds. In your answer, give the possible realizations of the cash flows, the probabilities of these events, and the payments made to junior and senior claims in each event. How much would a risk neutral investor be prepared to pay for the j and s claims. Is this more or less than they would pay for the underlying bonds? Explain. (10 points)

   (b) Now suppose there are *three* bonds, each as above. Explain how an intermediary can sell three prioritised claims (junior j, mezzanine m and senior s) against the possible cash flows from the three bonds. Give the possible realizations of cash flows, the probabilities of these events, and the payments made to junior and senior claims in each event. (10 points)

   (c) Now suppose there are *two pools each of two bonds* each as in part (a) above. Each pool has junior and senior claims. Explain how a financial intermediary can sell prioritised junior j_j and senior s_j claims to \$1 against the possible cash flows from a portfolio formed from the junior tranches j_1 and j_2 from each pool. What pattern of cash flows leads to senior claim in the second round of securitization being paid or not paid? Give the possible realizations of the cash flows, the probabilities of these events, and the payments made to the j_j and s_j claims from the second round of securitization. Would a risk neutral investor pay more for a senior claim in the first round of securitization (s_1 or s_2) or for a senior claim in the second round (s_j)? Explain. (15 points)

   (d) Now suppose there are two bonds as in part (a) except that the underlying bonds payments are perfectly positively correlated. Give the possible realizations of the cash flows, the probabilities of these events, and the payments made to junior and senior claims in each event. Would a risk neutral investor be prepared to pay a premium for senior claims? Explain. What if the underlying bond payments are instead perfectly *negatively* correlated, would your answers change? Would a risk averse investor view things differently? (5 points)

SOLUTIONS:

(a) The possible realizations and their probabilities are given in the table below. The calculations of the probabilities of each state use the fact that the probability of getting 1 is independent across bonds. In any state of the world where either bond pays out, we pay 1 to the senior claim. Only if both bonds pay out do we pay 1 to the junior claim. In this sense, the junior claim is the *residual claimant* to the cash flow from the package of bonds (like *equity*).

realization	$\{0,0\}$	$\{0,1\}$	$\{1,0\}$	$\{1,1\}$
probability	.0025	.0475	.0475	.9025
payment $\{j,s\}$	$\{0,0\}$	$\{0,1\}$	$\{0,1\}$	$\{1,1\}$

The probability of the junior claim being paid is therefore $\Pr(j=1) = .9025$ while the probability of the senior claim being paid is $\Pr(s=1) = .9025 + .0475 + .0475 = .9975$. A risk neutral investor would be willing to pay at most .9025 for the junior claim and at most .9975 for the senior claim. Therefore they would be willing to pay more for the senior claim than for one of the underlying bonds (due to the protection offered by the junior claim) but the junior claim is worth less than one of the underlying bonds.

(b) There are now $2^3 = 8$ possible realizations. These realizations are of 4 mutually exclusive types, illustrated in the table below: (i) no bonds pay, (ii) one out of three bonds pay, (iii) two out of three bonds pay, (iv) all bonds pay.

realization	$\{0,0,0\}$	$\{0,0,1\}$	$\cdots$	$\{1,1,0\}$	$\{1,1,1\}$
probability	.000125	.002375	$\cdots$	.045125	.857375
payment $\{j,m,s\}$	$\{0,0,0\}$	$\{0,0,1\}$	$\cdots$	$\{0,1,1\}$	$\{1,1,1\}$

The probability of the junior claim being paid is therefore just $\Pr(j=1) = .857375$, the probability of the mezzanine claim being paid is $\Pr(m=1) = .857375 + .045125 \times 3 = .99275$, while the probability of the senior claim being paid is $\Pr(s=1) = .99275 + .002375 \times 3 = .999875$. A risk neutral investor would be willing to pay at most .857375 for the junior claim, .99275 for the mezzanine and .999875 for the senior. Both the mezz and the senior are more valuable than the underlying bonds.

(c) Now we take the payments against the junior claims j_i for $i = 1, 2$ pools each of 2 bonds as in part (a). Below are the realizations, probabilities and cash flows in the second round of securitization.

realization $\{j_1, j_2\}$	$\{0,0\}$	$\{0,1\}$	$\{1,0\}$	$\{1,1\}$
probability	.0095	.088	.088	.8145
payment $\{j_j, s_j\}$	$\{0,0\}$	$\{0,1\}$	$\{0,1\}$	$\{1,1\}$

The probability of the junior claim in the second round being paid is therefore $\Pr(j_j = 1) = .8145$ while the probability of the senior claim in the second round being paid is $\Pr(s_j = 1) = .8145 + .088 \times 2 = .991$. In order for the junior claim in the first round to be paid out, there has to be no default in the pool on which that claim is written. So in order for the senior claim in the second round to be paid out, there has to be *no default in at least one* of the two pools of bonds. A risk neutral investor would pay at most .991 for a senior claim in the second round, i.e., less than the .9975 they'd be willing to pay for a senior claim in the first round. Although safer than the underlying bonds and the junior claims from the first round, the senior claim in the second round is still riskier than the senior claims in the first round.

(d) If the underlying bonds are perfectly *positively* correlated, then either both bonds pay out (with probability .95) or neither does (with probability .05). In this case there is no possibility of using prioritization (i.e., a *capital structure*) to protect a senior claim. Since there is no possibility of using prioritization, a risk neutral investor would pay at most .95 for a claim, the same as for the underlying bonds. If the bonds are instead perfectly *negatively* correlated, then a pool of two such bonds pays out 1 with probability 1.00 (since if one doesn't pay, the other does). Thus a claim to a pool of these two bonds can deliver 1 for sure and a risk neutral investor would be willing to pay 1 for such a claim (more than .95). Notice therefore that it is not correlation *per se* across the underlying bonds that destroys the ability to protect a senior claim, it is more specifically positive correlation that is the problem. Negative correlation across the underlying bonds makes it easier not harder to protect the senior claim (as always, at the cost of making the junior claim worth less).

In general, a risk averse investor will always need to be compensated for risk by being able to buy a security at a price lower than the risk neutral investor would be prepared to pay. How much of a discount depends on the curvature in their utility function. For CRRA utility with coefficient σ, the required discount is proportional to $\sigma/2$ times the variance of the cash flow [at least for small risks].

2. **Default risk in a portfolio of mortgages.** Consider a mortgage pool that consists of $i = 1, ..., n$ mortgages X_i. The X_i are IID *Bernoulli trials* which default $X_i = 1$ with probability p and do not default $X_i = 0$ with probability $1 - p$. The average default from a mortgage pool is p with variance $p(1 - p)$.

Now suppose that mortgage pools come in a variety of *types* each characterized by a particular value of the parameter p. These types of pools are distributed according to a probability density $f(p) > 0$ for $p \in [0, 1]$. Suppose also that we have a representative *portfolio* of these mortgage pools. Let $\bar{p}$ denote the portfolio average p, that is

$$\bar{p} \equiv \mathbb{E}[p] = \int_0^1 p f(p)\, dp$$

Notice that in this portfolio the variation in mortgage payments comes in two ways: *within* pool variation due to idiosyncratic realizations of X_i, and *between* pool variation due to differences in p. Conditional on p, the X_i within a pool are independent.

(a) Derive formulas for the portfolio average X_i, the portfolio variance of X_i and the correlation of two randomly chosen mortgages X_i and X_j from the portfolio. What is the correlation if all mortgage pools have $p = \bar{p}$? Explain. (15 points)

Hint: recall that for two random variables Y and Z, the *law of iterated expectations* says that $\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}(Y|Z)]$ and the *analysis of variance decomposition* gives $\text{Var}[Y] = \text{Var}[\mathbb{E}(Y|Z)] + \mathbb{E}[\text{Var}(Y|Z)]$.

Now let D_n denote the number of defaults within a given mortgage pool

$$D_n \equiv \sum_{i=1}^{n} X_i$$

and let D_n/n denote the corresponding *default rate*.

(b) Derive the portfolio average number of defaults and the portfolio variance of the number of defaults. What values do these statistics take when all mortgage pools have $p = \bar{p}$? What values do they take if $p = 1$ with probability $\bar{p}$ and $p = 0$ with probability $1 - \bar{p}$? Explain. (15 points)

(c) Derive the portfolio average default rate and the portfolio variance of the default rate. Consider the case where there are many mortgages within a given pool, i.e., where $n \to \infty$. In this case, how much of the variation in default rates comes from within a pool and how much from variation between pools? Explain. (15 points)

(d) Explain *intuitively* why the portfolio's frequency distribution of defaults approaches

$$\Pr\left(\frac{D_n}{n} < \theta\right) \to F(\theta) \qquad \text{as } n \to \infty$$

where $F(\cdot)$ is the cumulative probability distribution associated with $f(\cdot)$, that is

$$F(\theta) \equiv \int_0^\theta f(p)\, dp$$

What role does the distribution of mortgage pool types $f(p)$ play in making it possible for a financial intermediary to carve out tranches of differently-rated junior and senior claims to the mortgage payments? (15 points)

Solutions:

(a) Applying the law of iterated expectations we have

$$\mathbb{E}[X_i] = \mathbb{E}[\mathbb{E}(X_i \,|\, p)] = \mathbb{E}[p] = \bar{p}$$

and similarly using the ANOVA decomposition

$$\text{Var}[X_i] = \text{Var}[\mathbb{E}[X_i \,|\, p]] + \mathbb{E}[\text{Var}(X_i \,|\, p)]$$

$$= \text{Var}[p] + \mathbb{E}[p(1-p)]$$

Now recall that for any random variable Y the variance is $\text{Var}[Y] = \mathbb{E}[Y^2] - \mathbb{E}[Y]^2$ so

$$\text{Var}[X_i] = \text{Var}[p] + \mathbb{E}[p(1-p)]$$

$$= \mathbb{E}[p^2] - \mathbb{E}[p]^2 + \mathbb{E}[p] - \mathbb{E}[p^2]$$

$$= \bar{p}(1-\bar{p})$$

The correlation between X_i and X_j is defined by

$$\text{Corr}[X_i, X_j] \equiv \frac{\text{Cov}[X_i, X_j]}{\sqrt{\text{Var}[X_i]}\sqrt{\text{Var}[X_j]}}$$

Since $\text{Var}[X_i] = \text{Var}[X_j] = \bar{p}(1-\bar{p})$ this is just

$$\text{Corr}[X_i, X_j] = \frac{\text{Cov}[X_i, X_j]}{\bar{p}(1-\bar{p})}$$

Now recall that the covariance of any two random variables Y, Z is $\text{Cov}[Y, Z] = \mathbb{E}[YZ] - \mathbb{E}[Y]\mathbb{E}[Z]$ so that

$$\begin{aligned}
\text{Cov}[X_i, X_j] &= \mathbb{E}[X_i X_j] - \bar{p}^2 \\
&= \mathbb{E}[\mathbb{E}(X_i X_j \,|\, p)] - \bar{p}^2 \\
&= \mathbb{E}[\mathbb{E}(X_i \,|\, p)\mathbb{E}(X_j \,|\, p)] - \bar{p}^2 \\
&= \mathbb{E}[p^2] - \bar{p}^2 \\
&= \text{Var}[p]
\end{aligned}$$

where the second equality uses the law of iterated expectations and the third equality uses the fact that, conditional on p, the X_i and X_j are independent so that the (conditional) expectation of the product is the product of the (conditional) expectations. This tells us that the overall correlation between outcomes is ultimately driven by variation in p across pools. Also, perhaps surprisingly, this model can *only* give us *positively correlated* default outcomes (since $\text{Var}[p] \geq 0$). Since $\text{Var}[X_i] = \text{Var}[X_j] = \text{Var}[X]$ we can also write the correlation as

$$\text{Corr}[X_i, X_j] = \frac{\text{Var}[p]}{\text{Var}[X]}$$

In short, the correlation coefficient is the proportion of the unconditional variation in default outcomes that is accounted for by variation across pools. In the special case of all $p = \bar{p}$ (so that the distribution $f(p)$ is degenerate), there is no variation across mortgage pools and the correlation coefficient is zero. In this case, the overall correlation is the same as the within pool variation (since all pools are identical), namely zero.

(b) For the average number of defaults we have

$$\mathbb{E}[D_n] = \mathbb{E}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathbb{E}[X_i] = \sum_{i=1}^{n} \bar{p} = n\bar{p}$$

Now for the variance, a direction calculation gives

$$\text{Var}[D_n] = \text{Var}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \text{Var}[X_i] + \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \text{Cov}[X_i, X_j]$$

(this is the n variable generalization of the familiar formula that for any two random variables Y, Z we have $\text{Var}[Y + Z] = \text{Var}[Y] + \text{Var}[Z] + 2\text{Cov}[Y, Z]$). So using the results from part (a) we have

$$\text{Var}[D_n] = \sum_{i=1}^{n} \bar{p}(1-\bar{p}) + \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \text{Var}[p]$$

$$= n\bar{p}(1-\bar{p}) + n(n-1)\text{Var}[p]$$

This can be written equivalently in the form of an ANOVA decomposition

$$\text{Var}[D_n] = \text{Var}[\mathbb{E}[D_n \,|\, p]] + \mathbb{E}[\text{Var}(D_n \,|\, p)]$$

$$= \text{Var}[np] + \mathbb{E}[np(1-p)]$$

$$= \underbrace{n^2\text{Var}[p]}_{\text{between pool variation}} + \underbrace{n\mathbb{E}[p(1-p)]}_{\text{within pool variation}}$$

If there is no variation across mortgage pools $p = \bar{p}$, then clearly $\text{Var}[p] = 0$ and $\mathbb{E}[p(1-p) = \bar{p}(1-\bar{p})$ and the variance of D_n is the same as the variance of an individual pool, namely $n\bar{p}(1-\bar{p})$ (all the covariance terms are zero). In this case, all the variation is *within pool variation*. At the other extreme, if $p = 1$ with probability $\bar{p}$ or $p = 0$ with probability $1-\bar{p}$, then defaults are perfectly correlated within a given pool and so there is no variation within a pool. We can calculate directly that

$$\mathbb{E}[p(1-p)] = (1)\cdot(0)\cdot(\bar{p}) + (0)\cdot(1)\cdot(1-\bar{p}) = 0$$

(equivalently, $\mathbb{E}[p^2] = \bar{p}$) so that

$$\text{Var}[D_n] = n^2\text{Var}[p]$$

and indeed all the variation is *between pool variation*.

(c) For the average default rate we have

$$\mathbb{E}\left[\frac{D_n}{n}\right] = \frac{1}{n}\mathbb{E}[D_n] = \frac{1}{n}n\bar{p} = \bar{p}$$

and for the variance

$$\text{Var}\left[\frac{D_n}{n}\right] = \frac{1}{n^2}\text{Var}\left[D_n\right] = \frac{\bar{p}(1-\bar{p})}{n} + \frac{n-1}{n}\text{Var}[p]$$

As $n \to \infty$, the mean default rate remains $\bar{p}$ while the variance reduces to

$$\lim_{n\to\infty} \text{Var}\left[\frac{D_n}{n}\right] = \text{Var}[p]$$

In the limit of a large number n of mortgages, *all* the variation in default rates comes from variation in p between pools (and none from within a pool). If $p = \bar{p}$ for all pools, then the variance falls to zero since there is no longer any variation across pools.

(d) The basic idea is to use

$$\Pr\left(\frac{D_n}{n} < \theta\right) = \int_0^1 \Pr\left(\frac{D_n}{n} < \theta \,\middle|\, p\right) f(p)\, dp$$

and then to use a law of large numbers, $D_n/n \to p$ for $n \to \infty$, so that

$$\Pr\left(\frac{D_n}{n} < \theta \,\middle|\, p\right) \quad \to \quad \begin{cases} 0 & \text{if } p > \theta \\ 1 & \text{if } p < \theta \end{cases}, \qquad \text{as } n \to \infty$$

so that the frequency distribution of defaults is just given by $F(\cdot)$, namely

$$\begin{aligned} \Pr\left(\frac{D_n}{n} < \theta\right) &= \int_0^1 \Pr\left(\frac{D_n}{n} < \theta \,\middle|\, p\right) f(p)\, dp \\ &\to \int_0^\theta \left\{ \begin{matrix} 0 & \text{if } p > \theta \\ 1 & \text{if } p < \theta \end{matrix} \right\} f(p)\, dp, \quad \text{as } n \to \infty \\ &= \int_0^\theta f(p)\, dp = F(\theta) \end{aligned}$$

The ability to create senior claims depends crucially on there not being too much correlation. As the correlation increases, the value of senior claims falls and the value of junior claims rises until, in the limit of perfect correlation, the two are perfect substitutes. There is less correlation and hence more ability to create senior claims the more variation there is in p across pools, and this is a property of the $f(p)$ distribution.