Monetary Economics: Problem Set #5

Due Thursday October 9th in class

This problem set is marked out of 100 points. The weight given to each part is indicated below. Please contact me asap if you have any questions.

1. **Structured finance**. Suppose there are two bonds and that each pays \$1 cash or not. The probability of getting \$1 is 0.95 and is independent across bonds.

   (a) Explain how a financial intermediary can sell prioritised junior j and senior s claims to \$1 against the possible cash flows from a portfolio of these two bonds. In your answer, give the possible realizations of the cash flows, the probabilities of these events, and the payments made to junior and senior claims in each event. How much would a risk neutral investor be prepared to pay for the j and s claims. Is this more or less than they would pay for the underlying bonds? Explain. (10 points)

   (b) Now suppose there are *three* bonds, each as above. Explain how an intermediary can sell three prioritised claims (junior j, mezzanine m and senior s) against the possible cash flows from the three bonds. Give the possible realizations of cash flows, the probabilities of these events, and the payments made to junior and senior claims in each event. (10 points)

   (c) Now suppose there are *two pools each of two bonds* each as in part (a) above. Each pool has junior and senior claims. Explain how a financial intermediary can sell prioritised junior j_j and senior s_j claims to \$1 against the possible cash flows from a portfolio formed from the junior tranches j_1 and j_2 from each pool. What pattern of cash flows leads to senior claim in the second round of securitization being paid or not paid? Give the possible realizations of the cash flows, the probabilities of these events, and the payments made to the j_j and s_j claims from the second round of securitization. Would a risk neutral investor pay more for a senior claim in the first round of securitization (s_1 or s_2) or for a senior claim in the second round (s_j)? Explain. (15 points)

   (d) Now suppose there are two bonds as in part (a) except that the underlying bonds payments are perfectly positively correlated. Give the possible realizations of the cash flows, the probabilities of these events, and the payments made to junior and senior claims in each event. Would a risk neutral investor be prepared to pay a premium for senior claims? Explain. What if the underlying bond payments are instead perfectly *negatively* correlated, would your answers change? Would a risk averse investor view things differently? (5 points)

2. **Default risk in a portfolio of mortgages.** Consider a mortgage pool that consists of $i = 1, ..., n$ mortgages X_i. The X_i are IID *Bernoulli trials* which default $X_i = 1$ with probability p and do not default $X_i = 0$ with probability $1 - p$. The average default from a mortgage pool is p with variance $p(1 - p)$.

Now suppose that mortgage pools come in a variety of *types* each characterized by a particular value of the parameter p. These types of pools are distributed according to a probability density $f(p) > 0$ for $p \in [0, 1]$. Suppose also that we have a representative *portfolio* of these mortgage pools. Let $\bar{p}$ denote the portfolio average p, that is

$$\bar{p} \equiv \mathbb{E}[p] = \int_0^1 p f(p)\, dp$$

Notice that in this portfolio the variation in mortgage payments comes in two ways: *within* pool variation due to idiosyncratic realizations of X_i, and *between* pool variation due to differences in p. Conditional on p, the X_i within a pool are independent.

(a) Derive formulas for the portfolio average X_i, the portfolio variance of X_i and the correlation of two randomly chosen mortgages X_i and X_j from the portfolio. What is the correlation if all mortgage pools have $p = \bar{p}$? Explain. (15 points)

Hint: recall that for two random variables Y and Z, the *law of iterated expectations* says that $\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}(Y|Z)]$ and the *analysis of variance decomposition* gives $\text{Var}[Y] = \text{Var}[\mathbb{E}(Y|Z)] + \mathbb{E}[\text{Var}(Y|Z)]$.

Now let D_n denote the number of defaults within a given mortgage pool

$$D_n \equiv \sum_{i=1}^{n} X_i$$

and let D_n/n denote the corresponding *default rate*.

(b) Derive the portfolio average number of defaults and the portfolio variance of the number of defaults. What values do these statistics take when all mortgage pools have $p = \bar{p}$? What values do they take if $p = 1$ with probability $\bar{p}$ and $p = 0$ with probability $1 - \bar{p}$? Explain. (15 points)

(c) Derive the portfolio average default rate and the portfolio variance of the default rate. Consider the case where there are many mortgages within a given pool, i.e., where $n \to \infty$. In this case, how much of the variation in default rates comes from within a pool and how much from variation between pools? Explain. (15 points)

(d) Explain *intuitively* why the portfolio's frequency distribution of defaults approaches

$$\Pr\left(\frac{D_n}{n} < \theta\right) \to F(\theta) \qquad \text{as } n \to \infty$$

where $F(\cdot)$ is the cumulative probability distribution associated with $f(\cdot)$, that is

$$F(\theta) \equiv \int_0^\theta f(p)\, dp$$

What role does the distribution of mortgage pool types $f(p)$ play in making it possible for a financial intermediary to carve out tranches of differently-rated junior and senior claims to the mortgage payments? (15 points)