Monetary Economics: Problem Set #4
Solutions

This problem set is marked out of 100 points. The weight given to each part is indicated below. Please contact me asap if you have any questions.

1. **Government purchases in the new Keynesian model.** Consider a basic new Keynesian model with the following (log-linearised) equilibrium conditions: for consumption, a dynamic Euler equation

$$c_t = -\frac{1}{\sigma}(i_t - \mathbb{E}_t[\pi_{t+1}] - \rho) + \mathbb{E}_t[c_{t+1}]$$

for labor supply

$$w_t - p_t = \sigma c_t + \varphi n_t$$

the production function for firms

$$y_t = n_t$$

If prices were fully flexible, firms would set a constant markup over marginal cost. With sticky prices, there is a new Keynesian Phillips curve in terms of the output gap

$$\pi_t = \beta \mathbb{E}_t[\pi_{t+1}] + \kappa \tilde{y}_t$$

The government purchases a fraction τ_t of output each period with τ_t varying exogenously.

(a) Derive a log-linear version of the goods market clearing equation of the form $y_t = c_t + g_t$ where $g_t \equiv -\log(1-\tau_t)$. (5 points)

(b) Show that natural output is proportional to government purchases, $y_t^n = \Gamma g_t$, and give an explicit formula for the coefficient Γ. Does a fiscal expansion increase or decrease natural output when prices are fully flexible? By how much? Explain. (10 points)

Now assume that monetary policy is set according to the simple interest rate rule

$$i_t = \rho + \phi_\pi \pi_t + \phi_y \tilde{y}_t$$

and that government purchases g_t follow an AR(1) process with persistence $0 \leq \rho_g < 1$.

(c) Show that the output gap $\tilde{y}_t$ satisfies a dynamic IS curve of the form

$$\tilde{y}_t = -\frac{1}{\sigma}(i_t - \mathbb{E}_t[\pi_{t+1}] - r_t^n) + \mathbb{E}_t[\tilde{y}_{t+1}]$$

and derive a formula for the natural real rate r_t^n in terms of g_t. (5 points)

(d) Use the method of undetermined coefficients to solve for the response of the key endogenous variables —output, natural output, employment, inflation, interest rates— to an exogenous increase in government purchases g_t. Give economic intuition for your answers. (20 points)

(e) Explain how the response of output to government purchases depends on the monetary policy coefficient ϕ_π. Does a higher ϕ_π increase or decrease the impact effect of government purchases on output? Similarly, explain how the response of output to government purchases depends on the AR(1) persistence ρ_g. Does a more persistent process increase or decrease the impact effect of government purchases on output? Give economic intuition for your answers. (10 points)

SOLUTIONS:

(a) Since the government purchases a fraction τ_t of output, we have government purchases

$$G_t = \tau_t Y_t$$

and the goods market clearing condition in raw levels is

$$Y_t = C_t + G_t$$

so

$$(1 - \tau_t)Y_t = C_t$$

Taking logs of both sides

$$y_t = c_t + g_t$$

where, as suggested, $g_t \equiv -\log(1 - \tau_t)$ and as usual $y_t \equiv \log Y_t$, $c_t \equiv \log C_t$.

(b) With fully flexible prices each period, a firm sets its price as a constant markup over nominal marginal cost

$$P_t = \frac{\varepsilon}{\varepsilon - 1} W_t$$

or in logs

$$p_t = \mu + w_t$$

where $\mu \equiv \log(\varepsilon/(\varepsilon - 1)) > 0$ is the log markup. So the 'natural real wage' (for want of a better term) is a constant

$$w_t - p_t = -\mu$$

and so using the household's labor supply condition and the production function for firms, natural output y_t^n satisfies

$$\begin{aligned} -\mu &= w_t - p_t \\ &= \sigma c_t + \varphi n_t \\ &= \sigma(y_t^n - g_t) + \varphi y_t^n \\ &= (\sigma + \varphi) y_t^n - \sigma g_t \end{aligned}$$

Solving for y_t^n then gives

$$y_t^n = -\frac{1}{\sigma + \varphi}\mu + \frac{\sigma}{\sigma + \varphi} g_t \qquad (1)$$

When prices are fully flexible, the multiplier is

$$\Gamma \equiv \frac{\partial y_t^n}{\partial g_t} = \frac{\sigma}{\sigma + \varphi}$$

A fiscal expansion (an increase in g_t) increases the natural level of output. Why? Not for Keynesian demand-side reasons, that's for sure! The channel here is purely supply-side: an increase in g_t causes consumption c_t to fall which is a negative 'wealth effect' on labor supply, i.e., households feel 'poorer' (have higher marginal utility of consumption) and so work more at any given wage (an outward shift in the labor supply curve). That higher labor supply translates into higher natural output.

(c) The output gap is defined by

$$\tilde{y}_t \equiv y_t - y_t^n$$

and so using the resource constraint from part (a) and the expression for natural output from part (b) we have

$$\tilde{y}_t = c_t + g_t - y_t^n = c_t + g_t + \frac{1}{\sigma+\varphi}\mu - \frac{\sigma}{\sigma+\varphi} g_t$$

Rearrange this to express consumption in terms of the output gap and government purchases

$$c_t = \tilde{y}_t - \frac{1}{\sigma+\varphi}\mu + \left(\frac{\sigma}{\sigma+\varphi} - 1\right) g_t = \tilde{y}_t - \frac{1}{\sigma+\varphi}\mu - \frac{\varphi}{\sigma+\varphi} g_t$$

We now want to plug this into the dynamic Euler equation given in the problem

$$c_t = -\frac{1}{\sigma}(i_t - \mathbb{E}_t[\pi_{t+1}] - \rho) + \mathbb{E}_t[c_{t+1}]$$

Note that

$$\begin{aligned}\mathbb{E}_t[c_{t+1}] &= \mathbb{E}_t\Big[\tilde{y}_{t+1} - \frac{1}{\sigma+\varphi}\mu - \frac{\varphi}{\sigma+\varphi} g_{t+1}\Big] \\ &= \mathbb{E}_t[\tilde{y}_{t+1}] - \frac{1}{\sigma+\varphi}\mu - \frac{\varphi}{\sigma+\varphi}\mathbb{E}_t[g_{t+1}]\end{aligned}$$

and since g_t follows and AR(1) with persistence ρ_g we have

$$\mathbb{E}_t[g_{t+1}] = \rho_g g_t$$

so that

$$\mathbb{E}_t[c_{t+1}] = \mathbb{E}_t[\tilde{y}_{t+1}] - \frac{1}{\sigma+\varphi}\mu - \frac{\varphi}{\sigma+\varphi}\rho_g g_t$$

Plugging in for c_t and $\mathbb{E}_t[c_{t+1}]$ in the Euler equation then gives

$$\tilde{y}_t - \frac{1}{\sigma+\varphi}\mu - \frac{\varphi}{\sigma+\varphi} g_t = -\frac{1}{\sigma}(i_t - \mathbb{E}_t[\pi_{t+1}] - \rho) + \mathbb{E}_t[\tilde{y}_{t+1}] - \frac{1}{\sigma+\varphi}\mu - \frac{\varphi}{\sigma+\varphi}\rho_g g_t$$

Cancelling common terms and collecting things together

$$\tilde{y}_t = -\frac{1}{\sigma}(i_t - \mathbb{E}_t[\pi_{t+1}] - \rho) + \mathbb{E}_t[\tilde{y}_{t+1}] + \frac{\varphi}{\sigma+\varphi}(1-\rho_g) g_t \tag{2}$$

We can render this expression consistent with the dynamic IS curve given in the problem if we define the natural real rate r_t^n appropriately. In particular, define

$$r_t^n = \rho + \sigma\frac{\varphi}{\sigma+\varphi}(1-\rho_g) g_t \equiv \rho + \varphi_g g_t \tag{3}$$

so that the natural real rate is increasing in government purchases g_t. Then with this definition of the natural real rate, from equation (2) we have

$$\tilde{y}_t = -\frac{1}{\sigma}(i_t - \mathbb{E}_t[\pi_{t+1}] - \rho) + \mathbb{E}_t[\tilde{y}_{t+1}] + \frac{r_t^n - \rho}{\sigma}$$

or

$$\tilde{y}_t = -\frac{1}{\sigma}(i_t - \mathbb{E}_t[\pi_{t+1}] - r_t^n) + \mathbb{E}_t[\tilde{y}_{t+1}]$$

as required.

(d) Given the policy rule

$$i_t = \rho + \phi_\pi \pi_t + \phi_y \tilde{y}_t$$

the nominal rate less the natural rate is

$$i_t - r_t^n = \phi_\pi \pi_t + \phi_y \tilde{y}_t - \varphi_g g_t$$

where $\varphi_g > 0$ is the slope coefficient defined in (3) above. Plugging this into the dynamic IS curve we have the reduced system of equations that needs to be solved

$$\tilde{y}_t = -\frac{1}{\sigma}(\phi_\pi \pi_t + \phi_y \tilde{y}_t - \varphi_g g_t - \mathbb{E}_t[\pi_{t+1}]) + \mathbb{E}_t[\tilde{y}_{t+1}] \tag{4}$$

$$\pi_t = \beta \mathbb{E}_t[\pi_{t+1}] + \kappa \tilde{y}_t \tag{5}$$

Now guess the linear functions

$$\pi_t = \varphi_{\pi g} g_t$$
$$\tilde{y}_t = \varphi_{yg} g_t$$

so that

$$\mathbb{E}_t[\pi_{t+1}] = \varphi_{\pi g} \rho_g g_t$$
$$\mathbb{E}_t[\tilde{y}_{t+1}] = \varphi_{yg} \rho_g g_t$$

for some as-yet unknown coefficients that we need to solve for. Plugging these into the equations (4)-(5) and collecting terms gives

$$[\sigma(1-\rho_g) + \phi_y]\varphi_{yg} + (\phi_\pi - \rho_g)\varphi_{\pi g} = \varphi_g \tag{6}$$

$$(1 - \beta\rho_g)\varphi_{\pi g} = \kappa \varphi_{yg} \tag{7}$$

These are two equations to be solved for the two unknowns $\varphi_{yg}, \varphi_{\pi g}$. Solving, we get

$$\varphi_{\pi g} = \frac{\kappa}{(1-\beta\rho_g)[\sigma(1-\rho_g)+\phi_y] + \kappa(\phi_\pi - \rho_g)}\varphi_g \tag{8}$$

$$\varphi_{yg} = \frac{(1-\beta\rho_g)}{(1-\beta\rho_g)[\sigma(1-\rho_g)+\phi_y] + \kappa(\phi_\pi - \rho_g)}\varphi_g \tag{9}$$

If we impose the Taylor Principle $\phi_\pi > 1$ (i.e., the standard sufficient condition for a unique equilibrium, as discussed in class), then both of these coefficients are positive. Under this parameter assumption, then, an increase in government purchases g_t increases both inflation π_t and the output gap $\tilde{y}_t$. Since from equation (1) natural output y_t^n is increasing in g_t, output $y_t = \tilde{y}_t + y_t^n$ is also increasing in government purchases. Since from the production function $y_t = n_t$, labor n_t is also increasing in g_t. From the policy rule

$$i_t = \rho + \phi_\pi \pi_t + \phi_y \tilde{y}_t$$

and since both inflation and the output gap increase, so too does the nominal interest rate.

(e) Since the denominators of (8) and (9) are both increasing in ϕ_π, the response coefficients $\varphi_{\pi g}$ and φ_{yg} are both *decreasing* in ϕ_π. Moreover, since natural output does not depend on ϕ_π, the response of output itself is the same as the response of the output gap. So the response of output is also decreasing in ϕ_π. Intuitively, the more reactive the monetary authority is to inflation, the smaller is the effect of an increase in g_t on the output gap and hence the smaller is the rise in actual inflation. In this sense, the effectiveness of fiscal policy depends crucially on the monetary reaction.

The effects of ρ_g are a little more complicated. In particular, ρ_g also enters through the coefficient φ_g on the natural real rate, as defined in (3). Plugging this is and tidying the expression up gives

$$\varphi_{yg} = \frac{\sigma(1-\beta\rho_g)(1-\rho_g)}{\sigma(1-\beta\rho_g)(1-\rho_g) + (1-\beta\rho_g)\phi_y + \kappa(\phi_\pi - \rho_g)}(1-\Gamma) \tag{10}$$

To see the main effects of ρ_g intuitively, divide the numerator and denominator by $\sigma(1-\beta\rho_g)$ to write

$$\varphi_{yg} = \frac{1-\rho_g}{1-\rho_g+\psi}(1-\Gamma), \qquad \psi \equiv \frac{1}{\sigma}\Big[\phi_y + \frac{\kappa}{1-\beta\rho_g}(\phi_\pi - \rho_g)\Big] > 0$$

where $\psi > 0$ since $\phi_\pi > 1$ (and $1 > \rho_g$). Now first consider $\rho_g = 0$, a completely *transitory* stimulus. Then we have

$$0 < \varphi_{yg}\Big|_{\rho_g=0} = \frac{1}{1+\frac{1}{\sigma}\Big[\phi_y + \kappa\phi_\pi\Big]}(1-\Gamma) < 1-\Gamma$$

and so the multiplier, which is given by the effect on output itself, rather than the output gap, is greater than Γ but less than one. Moreover, at the other extreme, consider $\rho_g = 1$, a completely *permanent* stimulus (government purchases a random walk). Then we have

$$\varphi_{yg}\Big|_{\rho_g=1} = \frac{0}{0+\frac{1}{\sigma}\Big[\phi_y + \frac{\kappa}{1-\beta}(\phi_\pi - 1)\Big]}(1-\Gamma) = 0$$

in which case government purchases have zero effect on the output gap and hence the effect on output is simply Γ. In short, for $\rho_g = 0$ we have a multiplier less than one but larger than Γ while for $\rho_g = 1$ we have the lower multiplier Γ. Hence we expect the multiplier to be *decreasing* in ρ_g. The more persistent is the increase in government purchases, the larger is the expected inflation response and hence (if the Taylor principle is satisfied), the more vigorous is the monetary response.

For completeness, here is the calculation for general ρ_g. Dividing the numerator and denominator in (10) by $(1-\beta\rho_g)(1-\rho_g)$ we have

$$\varphi_{yg} = \frac{\sigma}{\sigma + \frac{1}{1-\rho_g}\phi_y + \kappa\frac{(\phi_\pi - \rho_g)}{(1-\beta\rho_g)(1-\rho_g)}}(1-\Gamma)$$

Hence the response coefficient φ_{yg} is decreasing in ρ_g if and only if

$$D(\rho_g) \equiv \frac{1}{1-\rho_g}\phi_y + \kappa\frac{(\phi_\pi - \rho_g)}{(1-\beta\rho_g)(1-\rho_g)}$$

is increasing in ρ_g. Differentiating $D(\rho_g)$ with respect to ρ_g gives, after some simplifications,

$$D'(\rho_g) = \frac{1}{(1-\rho_g)^2}\left[\phi_y + \frac{\kappa}{1-\beta\rho_g}\Big\{(\phi_\pi - \rho_g)(1+\beta-2\beta\rho_g) - (1-\rho_g)(1-\beta\rho_g)\Big\}\right]$$

which is certainly positive so long as the term in braces $\{\cdot\}$ is. This term is quadratic in ρ_g and can be rearranged to give the condition

$$Q(\rho_g) > 0 \Rightarrow D'(\rho_g) > 0$$

where

$$Q(\rho_g) \equiv \beta\rho_g^2 - 2\beta\phi_\pi\rho + (1+\beta)\phi_\pi - 1$$

Now observe that $Q(0) = (1+\beta)\phi_\pi - 1 > 0$ (since $\phi_\pi > 1$) but $Q(\rho_g)$ is decreasing in ρ_g, so it may seem that for high enough ρ_g we could get $Q(\rho_g) < 0$ which would be a pain. But in fact even at $\rho_g = 1$ we have $Q(1) = (1-\beta)(\phi_\pi - 1) > 0$ (again, since $\phi_\pi > 1$), so luckily we know that $Q(\rho_g) > 0$ for all $\rho_g \in [0,1]$. In turn this implies $D'(\rho_g) > 0$ for all ρ_g and so indeed, as conjectured, the response coefficient φ_{yg} is decreasing in ρ_g.

2. **Multipliers, the ZLB, and the duration of fiscal stimulus.** Consider a new Keynesian model with government purchases g_t and shocks Δ_t to the interest rate facing households

$$\tilde{y}_t = -\frac{1}{\sigma}(i_t + \Delta_t - \mathbb{E}_t[\pi_{t+1}] - r_t^n) + \mathbb{E}_t[\tilde{y}_{t+1}]$$

$$\pi_t = \beta\mathbb{E}_t[\pi_{t+1}] + \kappa\tilde{y}_t$$

Monetary policy is set according to an interest rate rule but also faces a *zero lower bound*

$$i_t = \max[0,\ \rho + \phi_\pi\pi_t + \phi_y\tilde{y}_t]$$

The natural real rate and natural output are given by

$$r_t^n = \rho - \sigma(1-\Gamma)\mathbb{E}_t[\Delta g_{t+1}], \qquad y_t^n = \Gamma g_t$$

The interest spread shock Δ_t can take on two values Δ_L, Δ_H with $\Delta_H = 0$ and $\Delta_L > 0$. The economy starts in the L state. With probability α it stays in the L state. With probability $1-\alpha$ it transitions to the H state. Once it enters the H ("normal") state it stays there forever. Suppose that $g_H = 0$. We are interested in calculating economic outcomes as a function of fiscal policy g_L in the L ("crisis") state.

(a) To begin with, suppose that $g_L = 0$. Solve for the equilibrium values of inflation, the output gap, and the nominal interest rate in the L state. (5 points)

(b) Now explain how an increase in government purchases to some $g_L > 0$ would affect inflation, expected inflation, output and the nominal and real interest rates in the L state. Explain how your answers depend on the size of the interest spread Δ_L. How large is the government purchases multiplier? Does this value depend on the the size of the fiscal stimulus? (15 points)

Now consider the possibility that government purchases g_L persist at an elevated level after the crisis has abated. To be specific, imagine that there are *three* states, L, S, H. The economy starts in the L state with $\Delta_L > 0$. Suppose the initial interest spread Δ_L is sufficiently high that the ZLB is binding in the L state. With probability α the economy transitions to the S state. In the S ("transitional") state, the crisis is over $\Delta_S = \Delta_H = 0$. In the S state, with probability λ the fiscal stimulus continues with government purchases $g_S = g_L > 0$. With probability $1-\lambda$ the fiscal stimulus ends with $g_S = g_H = 0$.

(c) Let $\pi_S, \tilde{y}_S, i_S$ denote the equilibrium values of inflation, the output gap and the nominal interest rate in the S state. Following the same logic as in part (a), solve for these equilibrium values as a function of the size of the fiscal stimulus $g_L > 0$. Explain how your answers depend on whether the fiscal stimulus continues $g_S = g_L$, or not $g_S = g_H$. (15 points)

(d) Using your results from part (c), now solve for the equilibrium values of inflation, the output gap and the nominal interest rate in the initial L state given that with probability λ the fiscal stimulus continues after the crisis has abated. How does the government purchases multiplier compare to the one you found in part (b)? How does the multiplier vary with the probability of the stimulus continuing? Explain. (15 points)

[*Hint*: this question is based on Woodford's article "Simple Analytics of the Government Expenditure Multiplier" *American Economic Journal: Macroeconomics*. **3**(1): 1–35, especially Section IV.B]

SOLUTIONS:

(a) From the new Keynesian Phillips curve in the low state we have

$$\pi_L = \frac{\kappa}{1-\alpha\beta}\tilde{y}_L$$

and the natural rate in the low state is

$$r_L^n = \rho + \sigma(1-\Gamma)(1-\alpha)\hat{g}_L$$

Plugging these into the IS curve in the low state we have

$$(1-\alpha)\tilde{y}_L = \frac{1}{\sigma}(r_L - i_L) + \frac{\alpha\kappa}{\sigma(1-\alpha\beta)}\tilde{y}_L + (1-\Gamma)(1-\alpha)\hat{g}_L$$

where $r_L \equiv \rho - \Delta_L$. So for a given i_L the output gap is

$$\tilde{y}_L = \vartheta_r(r_L - i_L) + \vartheta_g\hat{g}_L$$

with coefficients

$$\vartheta_r \equiv \frac{(1-\alpha\beta)}{\sigma(1-\alpha)(1-\alpha\beta)-\alpha\kappa} > 0$$

and

$$\vartheta_g \equiv \frac{\sigma(1-\alpha)(1-\alpha\beta)}{\sigma(1-\alpha)(1-\alpha\beta)-\alpha\kappa}(1-\Gamma) > 1-\Gamma > 0$$

But the nominal interest rate i_L is endogenous, given by the policy rule

$$\begin{aligned} i_L &= \max[0,\, \rho + \phi_\pi \pi_L + \phi_y \tilde{y}_L] \\ &= \max\left[0,\, \rho + \left(\phi_\pi \frac{\kappa}{1-\alpha\beta} + \phi_y\right) \tilde{y}_L\right] \\ &= \max\left[0,\, \rho + \left(\phi_\pi \frac{\kappa}{1-\alpha\beta} + \phi_y\right) (\vartheta_r(r_L - i_L) + \vartheta_g \hat{g}_L)\right] \end{aligned}$$

This is one nonlinear equation in one unknown, i_L. Once we have solved for i_L, we can then recover $\tilde{y}_L$ and π_L from the expressions above.

Mathematically, we are solving a problem of the form

$$x = f(x), \qquad f(x) \equiv \max[0, b - ax], \qquad x \geq 0$$

given two constants $a > 0$ and b. If $b > 0$ then this equation has a unique solution x^* and that solution lies on the positive branch, $x^* = b/(1+a) > 0$. But if $b < 0$ then there is a unique solution at $x^* = 0$. The size (and sign) of the intercept is determined by (i) the size of the shock, Δ_L and (ii) the size of the fiscal stimulus $\hat{g}_L$. The ZLB will bind if either the shock is large enough and/or the fiscal stimulus is too small.

(b) Specifically, suppose that $\hat{g}_L = 0$. Then the ZLB binds if

$$\rho + \left(\phi_\pi \frac{\kappa}{1-\alpha\beta} + \phi_y\right) \vartheta_r r_L < 0$$

or equivalently, whenever

$$r_L < r_L^* \equiv -\left(\left(\phi_\pi \frac{\kappa}{1-\alpha\beta} + \phi_y\right)\vartheta_r\right)^{-1} \rho < 0$$

Since $r_L = \rho - \Delta_L$, this is also equivalent to

$$\Delta_L > \Delta_L^* = \rho - r_L^* > 0$$

That is, the ZLB binds when the shock is large enough. Now if the ZLB binds (and continuing to assume $\hat{g}_L = 0$) then $i_L = 0$ and we have that the output gap is

$$\tilde{y}_L = \vartheta_r r_L$$

with inflation

$$\pi_L = \frac{\kappa}{1-\alpha\beta} \vartheta_r r_L$$

A small increase in government purchases $\hat{g}_L$ would cause output to rise to

$$\tilde{y}_L = \vartheta_r r_L + \vartheta_g \hat{g}_L$$

with a corresponding increase in inflation (and expected inflation) so long as the ZLB continues to bind. In this case, the government purchases multiplier is

$$\frac{dy_L}{d\hat{g}_L} = \frac{d\tilde{y}_L}{d\hat{g}_L} + \frac{dy_L^n}{d\hat{g}_L} = \vartheta_g + \Gamma > 1 - \Gamma + \Gamma > 1$$

If the increase in government purchases is large enough, however, the ZLB will cease to bind and the multiplier will fall back to $\Gamma < 1$. In short, output is a piecewise linear function of $\hat{g}_L$ with a slope that switches from above to below 1 at a critical “threshold” fiscal stimulus.

(c) The approach here follows the same steps as in part (a) above except that the transition probability α is replaced with λ and the interest spread in the transitional state is $\Delta_S = 0$. In particular, from the new Keynesian Phillips curve in the transitional state we have

$$\pi_S = \frac{\kappa}{1-\lambda\beta}\tilde{y}_S$$

and the natural rate in the transitional state is

$$r^n_S = \rho + \sigma(1-\Gamma)(1-\lambda)\hat{g}_S$$

Plugging these into the IS curve in the transitional state we then have

$$(1-\lambda)\tilde{y}_S = \frac{1}{\sigma}(\rho - i_S) + \frac{\lambda\kappa}{\sigma(1-\lambda\beta)}\tilde{y}_S + (1-\Gamma)(1-\alpha)\hat{g}_S$$

So for a given i_S the output gap is

$$\tilde{y}_S = \vartheta_{r,\lambda}(\rho - i_S) + \vartheta_{g,\lambda}\hat{g}_S$$

with coefficients

$$\vartheta_{r,\lambda} \equiv \frac{(1-\lambda\beta)}{\sigma(1-\lambda)(1-\lambda\beta)-\lambda\kappa} > 0$$

and

$$\vartheta_{g,\lambda} \equiv \frac{\sigma(1-\lambda)(1-\lambda\beta)}{\sigma(1-\lambda)(1-\lambda\beta)-\lambda\kappa}(1-\Gamma) > 1-\Gamma > 0$$

(i.e., the same as in part (a) with λ replacing α). Again, the nominal interest rate is found by solving

$$\begin{aligned} i_S &= \max[0,\ \rho + \phi_\pi\pi_S + \phi_y\tilde{y}_S] \\ &= \max\left[0,\ \rho + \left(\phi_\pi\frac{\kappa}{1-\lambda\beta} + \phi_y\right)\tilde{y}_S\right] \\ &= \max\left[0,\ \rho + \Psi\left(\vartheta_{r,\lambda}(\rho - i_S) + \vartheta_{g,\lambda}\hat{g}_S\right)\right] \end{aligned}$$

where $\Psi \equiv \phi_\pi\frac{\kappa}{1-\lambda\beta} + \phi_y > 0$. Once we have solved for i_S, we can then recover $\tilde{y}_S$ and π_S from the expressions above. Since the crisis is over, the intercept term is strictly positive,

$$\rho + \Psi\left(\vartheta_{r,\lambda}\rho + \vartheta_{g,\lambda}\hat{g}_S\right) > 0$$

so the solution is on the positive branch where the ZLB is slack. Hence the solution for i_S is given by

$$i_S = \rho + \frac{\Psi\vartheta_{g,\lambda}}{1+\Psi\vartheta_{r,\lambda}}\hat{g}_S$$

Plugging this into the expressions above for the output gap and inflation

$$\tilde{y}_S = \frac{\vartheta_{g,\lambda}}{1+\Psi\vartheta_{r,\lambda}}\hat{g}_S$$

and

$$\pi_S = \frac{\kappa}{1-\lambda\beta}\frac{\vartheta_{g,\lambda}}{1+\Psi\vartheta_{r,\lambda}}\hat{g}_S$$

Hence, if in the transitional state the fiscal stimulus does not continue, i.e., if $\hat{g}_S = \hat{g}_H = 0$, then $i_S = \rho$ and so in this case $\tilde{y}_S = 0$ and $\pi_S = 0$. In short, if the fiscal stimulus is over, then there is no difference between the transitional S state and the full end-of-crisis H state. But if the fiscal stimulus continues, i.e., if $\hat{g}_S = \hat{g}_L > 0$, then the interest rate is greater than ρ, there is a positive output gap $\tilde{y}_S > 0$ and positive inflation $\pi_S > 0$ (and positive expected inflation). In short, with an on-going active fiscal stimulus and away from the ZLB, monetary policy responds by increasing interest rates.

(d) (Sketch) We can now roll back to the crisis L period to see how the possibility of the fiscal stimulus continuing even after the crisis has abated affects the equilibrium in this earlier period. For example, we now have from the the new Keynesian Phillips curve in the L state

$$\pi_L = \beta[\alpha\pi_L + (1-\alpha)\pi_S] + \kappa\tilde{y}_L$$

so

$$\pi_L = \frac{\beta}{1-\alpha\beta}(1-\alpha)\pi_S + \frac{\kappa}{1-\alpha\beta}\tilde{y}_L$$

where π_S is given from part (c) above. The natural rate is

$$r_L^n = \rho - \sigma(1-\Gamma)(1-\alpha)(\hat{g}_S - \hat{g}_L)$$

which either simply equals ρ if the fiscal stimulus continues in the S state or is the same as in part (a) if the fiscal stimulus shuts off in the S state. Likewise, from the IS curve we then have

$$\tilde{y}_L = -\frac{1}{\sigma}(i_L + \Delta_L - \alpha\pi_L - (1-\alpha)\pi_S - r_L^n) + \alpha\tilde{y}_L + (1-\alpha)\tilde{y}_S$$

where again π_S and $\tilde{y}_S$ are determined as in part (c) above. To solve the model, we then need to determine the interest rate i_L from

$$i_L = \max[0,\ \rho + \phi_\pi\pi_L + \phi_y\tilde{y}_L]$$

It should be clear that if the fiscal stimulus does not continue, i.e., if $\hat{g}_S = \hat{g}_H = 0$, then everything reduces to being as in part (a) above. But if the fiscal stimulus does continue, so that $i_S > 0, \tilde{y}_S > 0$ and $\pi_S > 0$ then following the discussion in Woodford (pages 23–24) the size of the fiscal multiplier will be decreasing in the probability λ of the transitional state recurring (i.e., in the expected duration of the "excess" period of fiscal stimulus). Intuitively, this is because once inflation and the output gap increase in the transitional phase, monetary policy is off the ZLB and increases rates (via the interest rate rule coefficients ϕ_π, ϕ_y), i.e., monetary policy chokes off some of the fiscal policy-induced rise in demand. The more aggressive this monetary policy offsetting response, the smaller (or even negative) the fiscal policy multiplier.