Monetary Economics: Problem Set #3
Solutions

This problem set is marked out of 100 points. The weight given to each part is indicated below. Please contact me asap if you have any questions.

1. **Policy tradeoffs in the new Keynesian model.** Consider a new Keynesian model with output gap and inflation given by

$$\tilde{y}_t = -\frac{1}{\sigma}\left(i_t - \mathbb{E}_t[\pi_{t+1}] - \rho\right) + \mathbb{E}_t[\tilde{y}_{t+1}] \tag{1}$$

and

$$\pi_t = \beta \mathbb{E}_t[\pi_{t+1}] + \kappa \tilde{y}_t + x_t \tag{2}$$

where $\{x_t\}$ is an exogenous shock. Monetary policy is given by the interest rate rule

$$i_t = \rho + \phi_\pi \pi_t + v_t$$

where $\{v_t\}$ is an exogenous monetary policy shock and is independent of $\{x_t\}$.

(a) Explain in words the economic interpretation of equations (1) and (2). How could you interpret the x_t shock? (10 points)

To simplify the algebra, assume for the rest of this question that $\sigma = 1$ and that $\{x_t\}$ and $\{v_t\}$ are both IID white noise.

(b) Solve for equilibrium inflation π_t and the output gap $\tilde{y}_t$ in terms of the shocks v_t and x_t. (20 points)

(c) Explain how inflation, interest rates and the output gap respond to the v_t and x_t shocks. Give economic intuition for all your answers. (10 points)

(d) Suppose the central bank chooses the feedback coefficient ϕ_π to minimise the loss function

$$L = \mathrm{Var}[\pi_t] + \mathrm{Var}[\tilde{y}_t] \tag{3}$$

Solve for the value of ϕ_π that minimises this loss function. Is there a policy "trade-off" here? Explain how your answer depends on the parameter κ and on the variances of the shocks, $\mathrm{Var}[v_t]$ and $\mathrm{Var}[x_t]$. Give economic intuition for your answers. (15 points)

(e) Does the value of ϕ_π that minimises the loss function (3) satisfy the Taylor principle? Why or why not? (5 points)

SOLUTIONS:

(a) Putting aside the x_t shock, these equations are quite standard. Equation (1) is the (log-linear) intertemporal consumption Euler equation that results from household optimisation *plus* a simple goods market clearing condition of the form $c_t = y_t$. Equation (2) is the new Keynesian Phillips curve. This starts with an imperfectly competitive firm's optimal price-setting behaviour subject to a Calvo-style price-setting rigidity. This is then log-linearised and the terms reflecting real marginal cost are eliminated using the household's labor supply condition and approximate resource constraint so that it can be written in terms of the output gap $\tilde{y}_t^n \equiv y_t - y_t^n$ where natural output y_t^n is the level of output that would obtain in the same model but with perfectly flexible price-setting.

The x_t shock is a supply shock. An $x_t > 0$ represents higher inflation at any given level of the output gap, i.e., an *adverse* supply shock. In terms of micro-foundations, this could come from a shock to marginal cost, e.g., a shock to the relative price of an imported intermediate input like oil that the economy takes as given. (Such a shock increases costs but without lowering productivity).

(b) With the IID shocks, all the conditional expectations are zero and the system of equations dramatically simplifies. The new Keynesian Phillips curve reduces to a static aggregate supply relationship

$$\pi_t = \kappa \tilde{y}_t + x_t \tag{AS}$$

And using $\sigma = 1$ and rearranging, the Euler equation reduces to a static aggregate demand relationship

$$\pi_t = -\frac{1}{\phi_\pi}(\tilde{y}_t + v_t) \tag{AD}$$

Notice that the policy coefficient ϕ_π determines the slope of the AD curve. The more reactive is policy (the larger is ϕ_π), the flatter is the AD curve (see Figure 1).

These two equations can easily be solved for the output gap and inflation. Eliminating inflation between them

$$\kappa \tilde{y}_t + x_t = -\frac{1}{\phi_\pi}(\tilde{y}_t + v_t)$$

so

$$\tilde{y}_t = -\frac{\phi_\pi}{1+\kappa\phi_\pi}x_t - \frac{1}{1+\kappa\phi_\pi}v_t$$

and hence inflation is

$$\pi_t = \kappa \tilde{y}_t + x_t = -\kappa\left(\frac{\phi_\pi}{1+\kappa\phi_\pi}x_t + \frac{1}{1+\kappa\phi_\pi}v_t\right) + x_t = \frac{1}{1+\kappa\phi_\pi}x_t - \frac{\kappa}{1+\kappa\phi_\pi}v_t$$

(c) To summarise

$$\begin{aligned}
\tilde{y}_t &= -\frac{\phi_\pi}{1+\kappa\phi_\pi}x_t - \frac{1}{1+\kappa\phi_\pi}v_t \\
\pi_t &= \quad \frac{1}{1+\kappa\phi_\pi}x_t - \frac{\kappa}{1+\kappa\phi_\pi}v_t
\end{aligned}$$

with interest rates then given by

$$r_t = i_t = \rho + \phi_\pi \pi_t + v_t$$

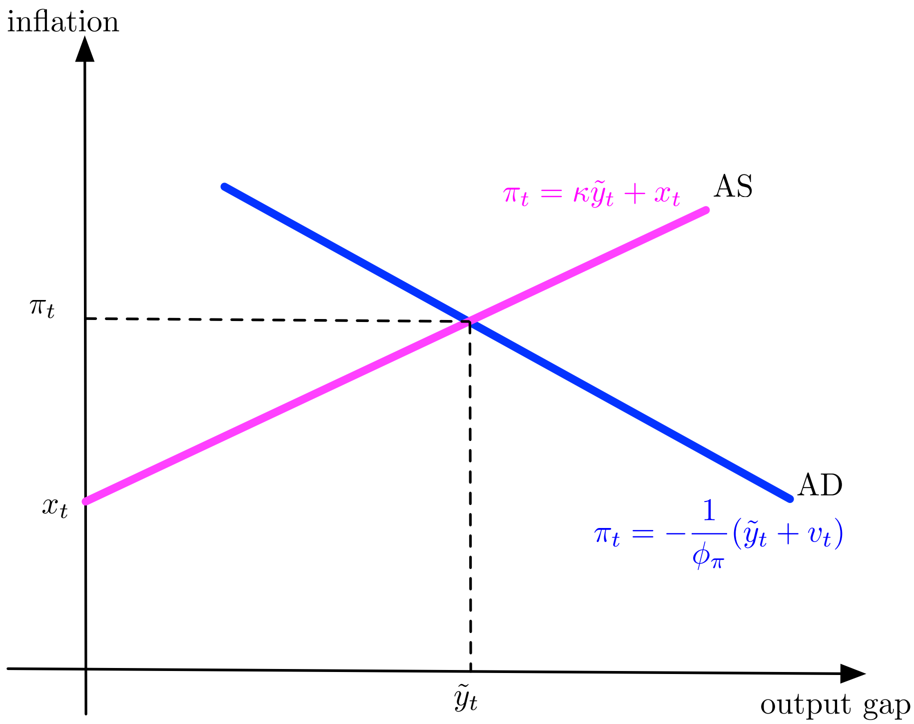

Figure 1: Static AS-AD model with demand shocks v_t and supply shocks x_t. Monetary policy sets the slope of the AD curve. A more reactive policy makes a flatter AD curve.

($r_t = i_t$ since expected inflation is zero). An adverse supply shock $x_t > 0$ increases inflation and decreases the output gap. This is a shift up of the aggregate supply relationship. A contractionary monetary policy shock $v_t > 0$ decreases inflation and decreases the output gap. This is a shift in of the aggregate demand relationship. (see Figure 2).

Since inflation is increasing in x_t, interest rates are too. What about the net effect of v_t?

$$\frac{\partial i_t}{\partial v_t} = 1 - \frac{\kappa\phi_\pi}{1+\kappa\phi_\pi} = \frac{1}{1+\kappa\phi_\pi}$$

which is positive (so long as ϕ_π is, i.e., so long as the AD curve slopes down).

(d) To begin with, the variances of inflation and the output gap are

$$\mathrm{Var}[\tilde{y}_t] = \left(\frac{\phi_\pi}{1+\kappa\phi_\pi}\right)^2 \mathrm{Var}[x_t] + \left(\frac{1}{1+\kappa\phi_\pi}\right)^2 \mathrm{Var}[v_t]$$
$$\mathrm{Var}[\pi_t] = \left(\frac{1}{1+\kappa\phi_\pi}\right)^2 \mathrm{Var}[x_t] + \left(\frac{\kappa}{1+\kappa\phi_\pi}\right)^2 \mathrm{Var}[v_t]$$

Adding these up and collecting terms, the loss function is

$$L = \left(\frac{1}{1+\kappa\phi_\pi}\right)^2 \left\{(\phi_\pi^2 + 1)\mathrm{Var}[x_t] + (1+\kappa^2)\mathrm{Var}[v_t]\right\}$$

If there were no supply shocks, $\mathrm{Var}[x_t] = 0$, then we would have the usual case from the lectures where there is no tradeoff and the optimal policy would have $\phi_\pi = +\infty$. Here

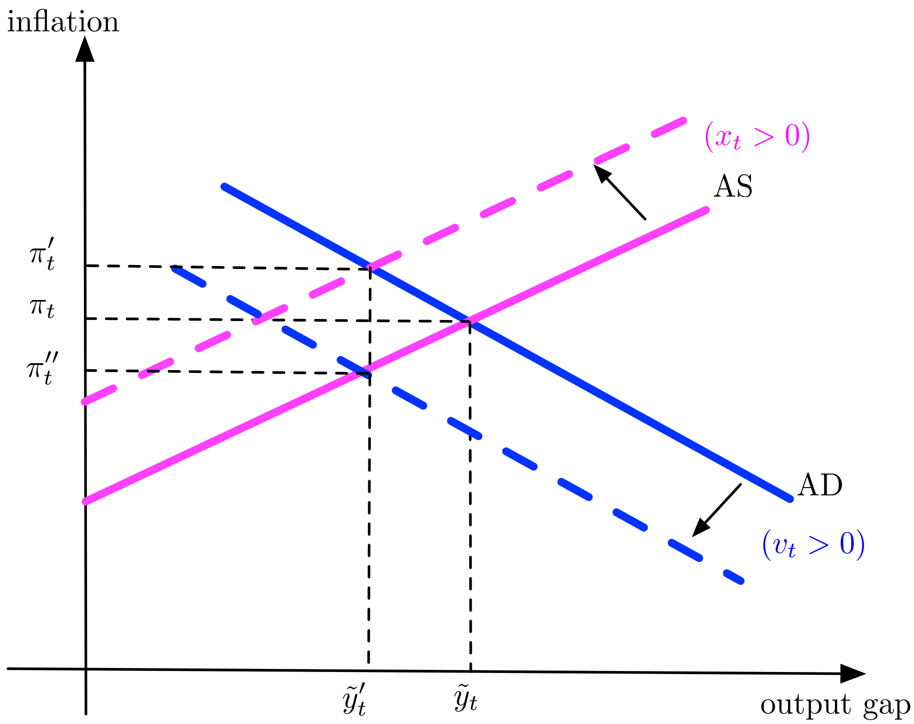

Figure 2: An adverse supply shock $x_t > 0$ shifts up of the AS curve, inflation rises and the output gap falls. A contractionary monetary policy shock $v_t > 0$ shifts in the AD curve, inflation and the output gap both fall.

that means the AD curve would be completely flat. Then inflation would be constant at zero and hence the variance of inflation would be zero too for any variance of the demand shocks v_t. However, if there are supply shocks then setting a high value of ϕ_π exposes the economy to large swings in the output gap and so when $\text{Var}[x_t] > 0$ there is a trade-off, as we can see in the formula for the loss function.

The first order condition for the optimal ϕ_π is

$$\frac{dL}{d\phi_\pi} = -2\frac{\kappa}{(1+\kappa\phi_\pi)^3}\left\{(\phi_\pi^2+1)\text{Var}[x_t] + (1+\kappa^2)\text{Var}[v_t]\right\} + \left(\frac{1}{1+\kappa\phi_\pi}\right)^2 2\phi_\pi\text{Var}[x_t] = 0$$

Cancelling the common factors and rearranging

$$(1+\kappa\phi_\pi)\phi_\pi\text{Var}[x_t] = \kappa\left\{(\phi_\pi^2+1)\text{Var}[x_t] + (1+\kappa^2)\text{Var}[v_t]\right\}$$

Cancelling more common terms (and thankfully the terms in ϕ_π^2) we get

$$\phi_\pi\text{Var}[x_t] = \kappa\text{Var}[x_t] + \kappa(1+\kappa^2)\text{Var}[v_t]$$

Hence

$$\phi_\pi^* = \kappa + \kappa(1+\kappa^2)\frac{\text{Var}[v_t]}{\text{Var}[x_t]} > \kappa > 0$$

The optimal coefficient ϕ_π^* is increasing in κ so that policy sets the slope of the AD curve $1/\phi_\pi^*$ to be *flatter* the *steeper* the slope of the AS curve. Intuitively, other things equal a steeper aggregate supply curve makes for more inflation volatility at the expense of output gap volatility and, since the preferences weigh each variance equally, policy counteracts that by reducing inflation volatility. The extent of this adjustment depends on κ and on the relative size of the fundamental shocks $\text{Var}[v_t]/\text{Var}[x_t]$.

(e) Yes, since $\phi_\pi^* > 0$. Here interest rates are given by

$$r_t = i_t = \rho + \phi_\pi^* \pi_t + v_t$$

Any positive coefficient will mean that nominal interest rates and hence real interest rates rise with inflation. This is because with the IID shocks, inflation expectations are constant. (Another way to put this is that any $\phi_\pi^* > 0$ makes the aggregate demand curve slope down here).

2. **Interest rate versus money supply rules.** Consider an economy again described by the equilibrium conditions

$$\tilde{y}_t = -\frac{1}{\sigma}(i_t - \mathbb{E}_t[\pi_{t+1}] - r_t^n) + \mathbb{E}_t[\tilde{y}_{t+1}]$$
$$\pi_t = \beta \mathbb{E}_t[\pi_{t+1}] + \kappa \tilde{y}$$

and now also a money demand equation of the form

$$m_t - p_t = y_t - \eta i_t, \qquad \eta > 0$$

where all variables are defined as usual. Both y_t^n and r_t^n evolve exogenously, independent of monetary policy. The central bank seeks to minimize a loss function of the form

$$L = \text{Var}[\pi_t] + \text{Var}[\tilde{y}_t]$$

(a) Explain how the optimal monetary policy outcomes can be implemented by an interest rate feedback rule. (10 points)

(b) Show that a constant money supply will generally not be optimal. (15 points)

(c) Derive a money supply rule that *would* implement the optimal monetary policy. (15 points)

SOLUTIONS:

(a) First observe that if we suppose an interest rate rule that does not depend on money m_t then this model can be solved without reference to the money demand condition $m_t - p_t = y_t - \eta i_t$. All that condition will do is enable us to determine, residually, the quantity of money consistent with the rest of the model.

Now consider an equilibrium outcome where $\pi_t = 0$ and $\tilde{y}_t = 0$ for all t. Since the unconditional variance of inflation and the output gap are both zero in such an equilibrium, the loss is $L = \text{Var}[\pi_t] + \text{Var}[\tilde{y}_t] = 0$ and clearly the monetary policy authority cannot do better than this. There are in general many interest rate rules that can implement this outcome. We discussed one example in class, a simple feedback rule of the form

$$i_t = r_t^n + \phi_\pi \pi_t + \phi_y \tilde{y}_t$$

If the coefficient on inflation ϕ_π is sufficiently high (i.e., the rule is sufficiently "reactive") then this rule will lead to a unique equilibrium outcome and in that outcome we will have $\pi_t = 0$ and $\tilde{y}_t = 0$. More specifically, following the lecture notes, in this setting the equilibrium dynamics are governed by

$$\begin{pmatrix} \tilde{y}_t \\ \pi_t \end{pmatrix} = \mathbf{A} \begin{pmatrix} \mathbb{E}_t[\tilde{y}_{t+1}] \\ \mathbb{E}_t[\pi_{t+1}] \end{pmatrix}$$

where

$$\mathbf{A} = \Omega \begin{pmatrix} \sigma & 1-\beta\phi_\pi \\ \sigma\kappa & \kappa + \beta(\sigma+\phi_y) \end{pmatrix}, \qquad \Omega \equiv \frac{1}{\sigma + \phi_y + \kappa\phi_\pi}$$

Now, when does $\mathbf{A}$ have *both* eigenvalues λ_1, λ_2 inside the unit circle (thus ensuring a unique outcome in the forward dynamics)? We have the following properties:

– **determinant**

$$\det(\mathbf{A}) = \Omega^2 \det \begin{pmatrix} \sigma & 1-\beta\phi_\pi \\ \sigma\kappa & \kappa + \beta(\sigma+\phi_y) \end{pmatrix} = \frac{\sigma\beta}{\sigma + \phi_y + \kappa\phi_\pi} = \lambda_1\lambda_2$$

– **trace**

$$\mathrm{tr}(\mathbf{A}) = \Omega \mathrm{tr} \begin{pmatrix} \sigma & 1-\beta\phi_\pi \\ \sigma\kappa & \kappa + \beta(\sigma+\phi_y) \end{pmatrix} = \frac{\sigma + \kappa + \beta(\sigma+\phi_y)}{\sigma + \phi_y + \kappa\phi_\pi} = \lambda_1 + \lambda_2$$

– **polynomial at unity**

$$p(1) = 1 - \mathrm{tr}(\mathbf{A}) + \det(\mathbf{A}) = \frac{\phi_y(1-\beta) + \kappa(\phi_\pi - 1)}{\sigma + \phi_y + \kappa\phi_\pi} = (1-\lambda_1)(1-\lambda_2)$$

Since the product $\lambda_1\lambda_2$ of the eigenvalues is positive, both eigenvalues must have the same sign (either both positive or both negative). Since the sum $\lambda_1+\lambda_2$ is positive, it then follows that both eigenvalues must be positive. Since $0 < \beta < 1$ the product of eigenvalues is less than one and so at least one of them is itself less than one. Therefore both eigenvalues are less than one if and only if $p(1) = (1-\lambda_1)(1-\lambda_2) > 0$, that is, if and only if

$$\phi_y(1-\beta) + \kappa(\phi_\pi - 1) > 0$$

Any interest sensitivity coefficient ϕ_π that satisfies this (e.g., $\phi_\pi > 1$) will suffice to ensure the unique equilibrium is $\pi_t = 0$ and $\tilde{y}_t = 0$ for all t and thus will minimise the loss.

(b) Consider the equilibrium generated by the interest rate rule given in part (a) above with ϕ_π such that this is the unique outcome. The money supply must then evolve according to

$$m_t = p_t + y_t - \eta i_t$$

or

$$\Delta m_t = \pi_t + \Delta y_t - \eta(i_t - i_{t-1})$$

But in this equilibrium, $\pi_t = 0$ and $\tilde{y}_t = 0$ so that the equilibrium nominal rate is just $i_t = r_t^n$ and equilibrium output is $y_t = y_t^n$ (since the output gap is zero). So money growth will be

$$\Delta m_t = \Delta y_t^n - \eta \Delta r_t^n$$

The right hand side here is exogenous and generally varying over time. Notice that this outcome *does not* depend on the parameters in the interest rate rule. Any interest rate rule that implements the optimal policy will imply money growth of this form. In all these situations, money growth would not generally be constant. Only in the very special case that Δy_t^n and Δr_t^n are both constants would we have constant money growth.

(c) The basic idea here is to make find a money growth rule that clears the money market at the same level of interest rates $i_t = r_t^n$ as would prevail in the equilibrium described above. Inverting the money market condition to write it in terms of the nominal rate

$$i_t = -\frac{1}{\eta}(m_t - p_t - y_t^n - \tilde{y}_t)$$

Plugging this into the dynamic IS curve gives

$$\tilde{y}_t = -\frac{1}{\sigma}\left(-\frac{1}{\eta}(m_t - p_t - y_t^n - \tilde{y}_t)) - \mathbb{E}_t\{\pi_{t+1}\} - r_t^n\right) + \mathbb{E}_t\{\tilde{y}_{t+1}\}$$

Now consider the specific *money supply rule*

$$m_t = p_t + y_t^n + \tilde{y}_t^n - \eta(r_t^n + \phi_\pi \pi_t + \phi_y \tilde{y}_t)$$

Plugging this into the dynamic IS curve gives

$$\begin{aligned}
\tilde{y}_t &= -\frac{1}{\sigma}\left(-\frac{1}{\eta}(p_t + y_t^n + \tilde{y}_t^n - \eta(r_t^n + \phi_\pi \pi_t + \phi_y \tilde{y}_t) - p_t - y_t^n - \tilde{y}_t)) - \mathbb{E}_t\{\pi_{t+1}\} - r_t^n\right) + \mathbb{E}_t\{\tilde{y}_{t+1}\} \\
&= -\frac{1}{\sigma}\left(-\frac{1}{\eta}(-\eta(r_t^n + \phi_\pi \pi_t + \phi_y \tilde{y}_t))) - \mathbb{E}_t\{\pi_{t+1}\} - r_t^n)\right) + \mathbb{E}_t\{\tilde{y}_{t+1}\} \\
&= -\frac{1}{\sigma}\left(-\frac{1}{\eta}(-\eta(r_t^n + \phi_\pi \pi_t + \phi_y \tilde{y}_t))) - \mathbb{E}_t\{\pi_{t+1}\} - r_t^n\right) + \mathbb{E}_t\{\tilde{y}_{t+1}\} \\
&= -\frac{1}{\sigma}\left(\phi_\pi \pi_t + \phi_y \tilde{y}_t - \mathbb{E}_t\{\pi_{t+1}\}\right) + \mathbb{E}_t\{\tilde{y}_{t+1}\}
\end{aligned}$$

This is one equilibrium condition in $\tilde{y}_t$ and π_t. The other condition is given by the new Keynesian Phillips curve. The equilibrium dynamics here satisfy the same system as in part (a), namely

$$\begin{pmatrix} \tilde{y}_t \\ \pi_t \end{pmatrix} = \mathbf{A} \begin{pmatrix} \mathbb{E}_t[\tilde{y}_{t+1}] \\ \mathbb{E}_t[\pi_{t+1}] \end{pmatrix}$$

where

$$\mathbf{A} = \Omega \begin{pmatrix} \sigma & 1 - \beta\phi_\pi \\ \sigma\kappa & \kappa + \beta(\sigma + \phi_y) \end{pmatrix}, \qquad \Omega \equiv \frac{1}{\sigma + \phi_y + \kappa\phi_\pi}$$

The coefficients ϕ_π etc here now refer to the coefficients in the money supply rule above. This money supply rule "mimics" the interest rate rule and, so long as

$$\phi_y(1 - \beta) + \kappa(\phi_\pi - 1) > 0$$

we have a unique equilibrium with $\pi_t = \tilde{y}_t = 0$ etc. In this equilibrium, money growth is as in part (b), namely

$$\Delta m_t = \Delta y_t^n - \eta \Delta r_t^n$$

and so is not generally constant.