Monetary Economics: Problem Set #1
Solutions

This problem set is marked out of 100 points. The weight given to each part is indicated below. Please contact me asap if you have any questions.

1. **A classical real economy.** A representative household maximises utility

$$U(C,N) = \frac{C^{1-\sigma}}{1-\sigma} - \frac{N^{1+\varphi}}{1+\varphi}, \qquad \sigma, \varphi > 0$$

subject to the budget constraint

$$PC \leq WN$$

A representative perfectly competitive firm has the linear production function

$$Y = AN$$

(a) Solve for the equilibrium levels of consumption C, labor N, output Y and the real wage W/P in terms of productivity A and the other parameters. (20 points)

(b) Suppose log productivity $a = \log A$ fluctuates randomly with variance $\text{Var}\{a\} = 1$. Calculate the variances of log consumption c, log labor n, log output y, and the log real wage $w-p$. Which of these variables is more volatile than productivity? Which is less volatile? Which of these variables is positively correlated with productivity? Which is negatively correlated? How do your answers depend on the preference parameters σ and φ? Give economic intuition for all your answers. (20 points)

Solutions:

(a) An equilibrium is characterized by household labor supply

$$-\frac{U_n}{U_c} = \frac{W}{P}$$

firm labor demand

$$\frac{W}{P} = A$$

and market clearing

$$C = Y = AN$$

With these preferences, the household marginal rate of substitution is $-U_n/U_c = N^{\varphi}C^{\sigma}$ and so combining the household labor supply and firm labor demand conditions we get

$$N^{\varphi}C^{\sigma} = \frac{W}{P} = A$$

Then substituting for C using the budget constraint we get one equation in one unknown, equilibrium employment N,

$$N^{\varphi}(AN)^{\sigma} = A$$

Solving this for equilibrium employment in terms of the exogenous level of productivity

$$N = A^{\frac{1-\sigma}{\varphi+\sigma}}$$

and hence equilibrium consumption and output are, from market clearing,

$$Y = C = AN = AA^{\frac{1-\sigma}{\varphi+\sigma}} = A^{\frac{\varphi+1}{\varphi+\sigma}}$$

and of course the real wage is just $W/P = A$.

(b) Taking logs of all the endogenous variables, we have

$$n = \frac{1-\sigma}{\varphi+\sigma}a$$

and

$$c = y = \frac{\varphi+1}{\varphi+\sigma}a$$

and the log real wage $w - p = a$. Now taking variances of all the endogenous variables

$$\text{Var}[n] = \left(\frac{1-\sigma}{\varphi+\sigma}\right)^2 \text{Var}[a] = \left(\frac{1-\sigma}{\varphi+\sigma}\right)^2$$

(since $\text{Var}[a] = 1$, by assumption) and similarly

$$\text{Var}[c] = \text{Var}[y] = \left(\frac{\varphi+1}{\varphi+\sigma}\right)^2$$

and the variance of the log real wage is 1 since the real wage is equal to productivity.

Employment is more volatile than productivity if and only if

$$\sigma < \frac{1-\varphi}{2}$$

and otherwise employment is less volatile than productivity. To understand this condition, first observe that with the linear production function labor demand is perfectly horizontal (i.e., firms are willing to hire any amount of labor at real wage $w - p = a$), so the *relative volatility* of employment is determined by the steepness of the labor supply curve. In particular, if the labor supply curve is sufficiently vertical (a sufficiently low Frisch elasticity of labor supply, or equivalently a sufficiently high φ), then employment does not vary much in response to productivity. What makes for φ sufficiently high? Since $\sigma > 0$, any $\varphi > 1$ will make employment less volatile than productivity.

Employment is perfectly positively *correlated* with productivity if $\sigma < 1$ and perfectly negatively correlated with productivity if $\sigma > 1$. If $\sigma = 1$, then employment is constant and hence uncorrelated with productivity. The intuition here is that if $\sigma < 1$, the substitution effect in labor supply dominated and employment rises with the real wage which rises with productivity. If $\sigma > 1$, the income effect of a higher real wage dominates and employment falls when the real wage rises. If $\sigma = 1$ (i.e., log utility) then the income and substitution effects exactly balance.

Consumption and output are always perfectly positively correlated with productivity. They are more volatile than productivity if

$$\frac{\varphi + 1}{\varphi + \sigma} > 1$$

equivalently, if

$$1 > \sigma$$

Intuitively, if high productivity calls forth more labor supply (the substitution effect dominates) then output and consumption will fluctuate more than productivity.

2. **Real interest rates in the classical model.** Consider a classical model with the following (log-linearised) household optimality conditions:

$$c_t = -\frac{1}{\sigma}(r_t - \rho) + \mathbb{E}_t[c_{t+1}], \qquad \rho, \sigma > 0 \tag{1}$$

and

$$\sigma c_t + \varphi n_t = w_t - p_t, \qquad \varphi > 0 \tag{2}$$

Perfectly competitive firms choose labor demand to maximise profits subject to the (log-linear) production function

$$y_t = a_t + n_t$$

where $\{a_t\}$ is log productivity which follows an AR(1) process

$$a_{t+1} = \rho_a a_t + \epsilon_{t+1}, \qquad 0 \le \rho_a < 1$$

where $\{\epsilon_t\}$ is an IID white noise shock.

(a) Explain in words the economic interpretation of equations (1) and (2). (15 points)

(b) Solve for the equilibrium levels of (log) consumption c_t, employment n_t and output y_t in terms of productivity a_t and the exogenous parameters. Briefly explain the effects of a positive productivity shock on each of these endogenous variables. Give intuition for all your answers. (15 points)

(c) Solve for the equilibrium real interest rate r_t in terms of the productivity process and other parameters. Does an increase in productivity increase or decrease the real interest rate? Does a higher value of σ increase or decrease the sensitivity of the real interest rate to a productivity shock? Explain. (15 points)

(d) Suppose the productivity process is instead a random walk with drift $\gamma > 0$

$$a_{t+1} = \gamma + a_t + \epsilon_{t+1}$$

Does an increase in productivity increase or decrease the real interest rate? Does a higher value of σ increase or decrease the sensitivity of the real interest rate to a productivity shock? Explain the differences, if any, between your answers for parts (c) and (d). (15 points)

SOLUTIONS:

(a) Equation (1) is the representative household's log-linearised intertemporal Euler equation that governs consumption smoothing. Implicitly the representative household is being assumed to have a separable period utility function, since only consumption (and not, say, labor) enters the Euler equation. Moreover utility from consumption is of the CRRA form with coefficient σ. Notice that a higher real rate r_t induces a lower level of consumption c_t (other things equal), while a real interest rate greater than the constant rate of time preference, $r_t > \rho$, induces consumption that is growing in expectation

$$\mathbb{E}_t[\Delta c_{t+1}] = \frac{r_t - \rho}{\sigma}$$

with interest sensitivity given by $1/\sigma$, i.e., by the constant intertemporal elasticity of substitution. Equation (2) is the representative household's labor supply condition, equating the marginal rate of substitution between labor and consumption to the real wage. The particular log-linear form here again indicates a period utility function that is separable between consumption and labor with curvature over consumption again given by σ and curvature over labor given by φ. In this context, $1/\varphi$ is the so-called *Frisch* (or "λ-constant") elasticity of labor supply. This terminology comes from writing the labor supply condition

$$n_t = \frac{1}{\varphi}(w_t - p_t) - \frac{\sigma}{\varphi}c_t$$

so that $1/\varphi$ is the elasticity of labor supply with respect to the real wage *holding fixed the marginal utility of consumption*, i.e., ignoring the wealth effect of changing real wages.

(b) Firm labor demand is governed by

$$w_t - p_t = a_t$$

so that the real wage is equated to the marginal product of labor. Then using the household labor supply condition

$$\sigma c_t + \varphi n_t = w_t - p_t = a_t$$

The market clearing condition is $c_t = y_t$ and from the production function $y_t = a_t + n_t$ so that employment solves

$$\sigma(a_t + n_t) + \varphi n_t = a_t$$

or

$$n_t = \frac{1-\sigma}{\varphi + \sigma}a_t$$

As usual, employment responds positively to an increase in productivity if the substitution effect dominates the income effect, $\sigma < 1$, and responds negatively to a productivity shock if the income effect dominates the substitution effect, $\sigma > 1$. With the solution for employment in hand, output and consumption are

$$c_t = y_t = a_t + n_t = \frac{\varphi + 1}{\varphi + \sigma}a_t$$

Both consumption and output respond positively to an increase in productivity, the response is more than 1-for-1 if employment increases and less than 1-for-1 if employment decreases but the effect is always positive.

(c) Let ψ denote the elasticity of output with respect to productivity, that is

$$\psi \equiv \frac{\varphi + 1}{\varphi + \sigma}$$

so that $c_t = y_t = \psi a_t$. Then from the consumption Euler equation the real interest rate r_t satisfies

$$\psi a_t = -\frac{1}{\sigma}(r_t - \rho) + \mathbb{E}_t[\psi a_{t+1}]$$

Solving for the real interest rate r_t we get

$$r_t = \rho + \sigma\psi\mathbb{E}_t[\Delta a_{t+1}]$$

where $\Delta a_{t+1} = a_{t+1} - a_t$ is productivity growth. Using the AR(1) process for productivity, the conditional expectation of productivity is $\mathbb{E}_t[a_{t+1}] = \rho_a a_t$ so that

$$r_t = \rho + \sigma\psi\mathbb{E}_t[\Delta a_{t+1}] = \rho + \sigma\psi(\rho_a - 1)a_t$$

Since $\rho_a < 1$, the real interest rate *falls* when productivity increases. The intuition for this is that when productivity increases above average, it is then expected to fall back towards its mean value (the AR(1) is *mean-reverting*) so that expected consumption growth is negative. In order for consumption growth to slow, the real interest rate falls. For a given ψ, a higher value of σ makes real interest rates *more* sensitive to consumption growth, precisely because higher σ makes consumers *less* interest sensitive. An increase in σ also has the effect of reducing ψ, making the level of consumption less sensitive to productivity, but the net effect is given by the product

$$\sigma\psi = (\varphi + 1)\frac{\sigma}{\varphi + \sigma}$$

which is clearly increasing in σ.

(d) The real interest rate is still given by

$$r_t = \rho + \sigma\psi\mathbb{E}_t[\Delta a_{t+1}]$$

But now expected productivity growth is $\mathbb{E}_t[\Delta a_{t+1}] = \gamma$, so that the real interest rate is a constant

$$r = \rho + \sigma\psi\gamma$$

The real interest rate is a constant because the current level of productivity a_t does not change the forecast of productivity growth between t and $t+1$, i.e., there is no mean-reversion in productivity. The effects of σ are essentially the same as in part (b) in that the response of the real rate r to growth is given by the product $\sigma\psi = (\varphi + 1)\sigma/(\varphi + \sigma)$ which is increasing in σ so that the real interest rate responds more to growth when consumers are less interest sensitive.