Monetary Economics

Lecture 9: monetary policy in the new Keynesian model, part two

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This class

- Monetary policy in the new Keynesian model, part two
  - equilibrium stability and uniqueness
  - implementation of optimal policy

- Reading: Gali, chapter 4 section 4.3

This class

1- Equilibrium stability and uniqueness

- forwards and backwards dynamics
- eigenvalues etc

2- Implementation of optimal policy

- passive rules
- active (feedback) rules

Stability of dynamical systems

- Consider scalar equation

$$x_{t+1} = \rho x_t$$

- Suppose you have given initial condition x_0. Solve *backwards*, stable solution if and only if

$$|\rho| < 1 \qquad \text{such that} \qquad x_t = \rho^t x_0 \to 0$$

- But if no given initial condition ("jump" variable), solve *forwards*

$$x_t = a x_{t+1}, \qquad a \equiv \rho^{-1}$$

stable solution if and only if

$$|a| < 1 \qquad \text{such that} \qquad x_t = 0 \qquad \text{for all } t$$

- Note: stability depends on forwards or backwards dynamics

Stability of dynamical systems

- Now consider system

$$\mathbf{x}_t = \mathbf{A}\mathbf{x}_{t+1}, \qquad \text{perhaps some } \mathbf{x}_0 \text{ given}$$

- For unique solution, need same number of stable variables in *forward dynamics* as there are missing initial conditions

- Example: optimal growth model

  - two dynamic variables (capital and consumption)
  - one given initial condition (capital stock)
  - one stable variable in the forward dynamics (consumption)

Stability of dynamical systems

- Consider 2-by-2 dynamical system

$$\mathbf{x}_t = \mathbf{A}\mathbf{x}_{t+1}, \qquad \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

- If system is *diagonal*, then

$$\mathbf{A}^t = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix}^t = \begin{pmatrix} a_{11}^t & 0 \\ 0 & a_{22}^t \end{pmatrix}$$

(stability determined by diagonal coefficients)

- For large class of "regular" matrices, can uncouple or *diagonalize*

$$\mathbf{A} = \mathbf{Q}\mathbf{D}\mathbf{Q}^{-1}$$

where $\mathbf{D}$ is diagonal matrix of *eigenvalues*

Stability of dynamical systems

- Then do change of variables ("rotation")

$$\mathbf{z}_t \equiv \mathbf{Q}^{-1}\mathbf{x}_t$$

so that

$$\mathbf{z}_t = \mathbf{D}\mathbf{z}_{t+1}$$

- Stability properties of $\mathbf{x}_t$ equivalent to those of $\mathbf{z}_t$

$\Rightarrow$ study eigenvalues of coefficient matrix $\mathbf{A}$

Eigenvalues

- For a square matrix $\mathbf{A}$ a scalar λ is an *eigenvalue* if and only if

$$\mathbf{B} \equiv \mathbf{A} - \lambda \mathbf{I}$$

is *singular*

$\Leftrightarrow$ there are solutions to $\mathbf{Bx} = \mathbf{0}$ other than $\mathbf{x} = \mathbf{0}$
$\Leftrightarrow$ the *determinant* of $\mathbf{B}$ is zero

- For a 2-by-2 matrix, simple formula for determinant

$$\det(\mathbf{B}) = \det \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = b_{11}b_{22} - b_{12}b_{21}$$

- Therefore

$$\begin{aligned} \det(\mathbf{A} - \lambda \mathbf{I}) &= \det \begin{pmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{pmatrix} \\ &= (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} \\ &= \lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21} \end{aligned}$$

Eigenvalues

- So for a 2-by-2 matrix, the eigenvalues solve a quadratic equation

$$p(\lambda) \equiv \lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21} = 0$$

(the *characteristic polynomial*)

- Two roots. From the quadratic formula

$$\lambda_1, \lambda_2 = \frac{(a_{11} + a_{22}) \pm \sqrt{(a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21})}}{2}$$

- Roots may be real or complex, repeated or distinct
  - if roots repeated, diagonalization a bit more complicated

Key properties of eigenvalues

- Determinant of n-by-n matrix is product of eigenvalues

$$\det(\mathbf{A}) = \prod_{i=1}^{n} \lambda_i, \qquad \det(c\mathbf{A}) = c^n \det(\mathbf{A})$$

- Trace of n-by-n matrix is sum of eigenvalues

$$\operatorname{tr}(\mathbf{A}) \equiv \sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} \lambda_i, \qquad \operatorname{tr}(c\mathbf{A}) = c\operatorname{tr}(\mathbf{A})$$

- So for 2-by-2 matrix can write characteristic polynomial

$$p(\lambda) = \lambda^2 - \operatorname{tr}(\mathbf{A})\lambda + \det(\mathbf{A}) = (\lambda - \lambda_1)(\lambda - \lambda_2)$$

and so relative magnitudes of eigenvalues determined by

$$p(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)$$

therefore

$$p(1) = (1-\lambda_1)(1-\lambda_2) > 0 \Leftrightarrow \text{both eigenvalues on same side of } +1$$

Back to the new Keynesian model

- Shut down shocks to focus on deterministic dynamics (straightforward to add back in). Non-policy block of the model

$$\tilde{y}_t = -\frac{1}{\sigma}\left(i_t - \mathbb{E}_t\{\pi_{t+1}\} - r_t^n\right) + \mathbb{E}_t\{\tilde{y}_{t+1}\}$$

and

$$\pi_t = \beta \mathbb{E}_t\left\{\pi_{t+1}\right\} + \kappa \tilde{y}_t$$

- Optimal policy entails

$$\begin{aligned} \pi_t &= 0 \\ \tilde{y}_t &= 0 \\ i_t &= r_t^n \end{aligned}$$

- Let's try and implement this with passive $i_t = r_t^n$ rule

System of equations

- For passive $i_t = r_t^n$ policy, system of equations just

$$\begin{pmatrix} 1 & 0 \\ -\kappa & 1 \end{pmatrix} \begin{pmatrix} \tilde{y}_t \\ \pi_t \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{\sigma} \\ 0 & \beta \end{pmatrix} \begin{pmatrix} \mathbb{E}_t[\tilde{y}_{t+1}] \\ \mathbb{E}_t[\pi_{t+1}] \end{pmatrix}$$

- Matrix algebra trick (for 2-by-2 matrices)

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

- With this trick

$$\begin{pmatrix} 1 & 0 \\ -\kappa & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ \kappa & 1 \end{pmatrix}$$

- Therefore

$$\begin{pmatrix} \tilde{y}_t \\ \pi_t \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{\sigma} \\ \kappa & \beta + \frac{\kappa}{\sigma} \end{pmatrix} \begin{pmatrix} \mathbb{E}_t[\tilde{y}_{t+1}] \\ \mathbb{E}_t[\pi_{t+1}] \end{pmatrix}$$

Stability in the new Keynesian model

- In short

$$\begin{pmatrix} \tilde{y}_t \\ \pi_t \end{pmatrix} = \mathbf{A} \begin{pmatrix} \mathbb{E}_t[\tilde{y}_{t+1}] \\ \mathbb{E}_t[\pi_{t+1}] \end{pmatrix}$$

- Stability properties depend crucially on eigenvalues of **A**
  - two endogenous dynamic variables
  - no given initial conditions (both are "jump" variables)
  - for uniqueness, need both eigenvalues of **A** *inside* unit circle (in forward dynamics)

Passive interest rate rule $i_t = r_t^n$

- Determinant

$$\det(\mathbf{A}) = \det\begin{pmatrix} 1 & \frac{1}{\sigma} \\ \kappa & \beta + \frac{\kappa}{\sigma} \end{pmatrix} = \beta = \lambda_1 \lambda_2$$

- Trace

$$\mathrm{tr}(\mathbf{A}) = \mathrm{tr}\begin{pmatrix} 1 & \frac{1}{\sigma} \\ \kappa & \beta + \frac{\kappa}{\sigma} \end{pmatrix} = 1 + \beta + \frac{\kappa}{\sigma} = \lambda_1 + \lambda_2$$

- Polynomial at unity

$$p(1) = 1 - \mathrm{tr}(\mathbf{A}) + \det(\mathbf{A}) = -\frac{\kappa}{\sigma} < 0$$

- Implications

(i) product positive, so both eigenvalues have same sign
(ii) sum is positive, therefore from (i) both positive
(iii) polynomial $p(1) < 0$, eigenvalues not on same side of $+1$

$$0 < \lambda_1 < 1 < \lambda_2$$

Passive interest rate rule $i_t = r_t^n$

- Coefficient matrix $\mathbf{A}$ has $\lambda_1 < 1$ and $\lambda_2 > 1$
- Passive policy with $i_t = r_t^n$ has $\pi_t = \tilde{y}_t = 0$ as *an* equilibrium
- But there are *multiple* equilibria
  - one dimensional degree of indeterminacy
  - no reason to believe optimal outcome will emerge
- What about other rules? Can they 'reliably' implement optimum?

Feedback rule

- Consider interest rate rule with feedback

$$i_t = r_t^n + \phi_\pi \pi_t + \phi_y \tilde{y}_t$$

- Now deterministic dynamics governed by

$$\begin{pmatrix} \tilde{y}_t \\ \pi_t \end{pmatrix} = \mathbf{A} \begin{pmatrix} \mathbb{E}_t[\tilde{y}_{t+1}] \\ \mathbb{E}_t[\pi_{t+1}] \end{pmatrix}$$

where

$$\mathbf{A} = \Omega \begin{pmatrix} \sigma & 1 - \beta\phi_\pi \\ \sigma\kappa & \kappa + \beta(\sigma + \phi_y) \end{pmatrix}, \qquad \Omega \equiv \frac{1}{\sigma + \phi_y + \kappa\phi_\pi}$$

- When does **A** have both eigenvalues inside unit circle?

Feedback rule

- Determinant

$$\det(\mathbf{A}) = \Omega^2 \det \begin{pmatrix} \sigma & 1-\beta\phi_\pi \\ \sigma\kappa & \kappa+\beta(\sigma+\phi_y) \end{pmatrix} = \frac{\sigma\beta}{\sigma+\phi_y+\kappa\phi_\pi} = \lambda_1\lambda_2$$

- Trace

$$\mathrm{tr}(\mathbf{A}) = \Omega \mathrm{tr} \begin{pmatrix} \sigma & 1-\beta\phi_\pi \\ \sigma\kappa & \kappa+\beta(\sigma+\phi_y) \end{pmatrix} = \frac{\sigma+\kappa+\beta(\sigma+\phi_y)}{\sigma+\phi_y+\kappa\phi_\pi} = \lambda_1 + \lambda_2$$

- Polynomial at unity

$$p(1) = 1 - \mathrm{tr}(\mathbf{A}) + \det(\mathbf{A}) = \frac{\phi_y(1-\beta)+\kappa(\phi_\pi - 1)}{\sigma+\phi_y+\kappa\phi_\pi}$$

- Implications

  (i) product positive, so both eigenvalues have same sign
  (ii) sum is positive, therefore from (i) both positive
  (iii) since $\beta < 1$, product is < 1, therefore at least one is < 1
  (iv) therefore both eigenvalues < 1 if and only if $p(1) > 0$

- Equivalently, if and only if

$$\phi_y(1-\beta) + \kappa(\phi_\pi - 1) > 0$$

- A sufficient condition is for $\phi_\pi > 1$

- Interest rate response needs to be "*sufficiently reactive*"

Taylor principle

- Intuition. In steady state

$$\frac{\partial i}{\partial \pi} = \phi_\pi + \phi_y \frac{\partial \tilde{y}}{\partial \pi}, \qquad \frac{\partial \tilde{y}}{\partial \pi} = \frac{1-\beta}{\kappa}$$

- Therefore

$$\frac{\partial i}{\partial \pi} > 1 \qquad \Leftrightarrow \qquad \phi_\pi + \phi_y \frac{1-\beta}{\kappa} > 1$$

- This is the same as our condition from the polynomial at unity

- Real interest rate rises in response to inflation

Equilibrium outcome

- Suppose $\phi_\pi > 1$ so that condition is satisfied

- In equilibrium, both endogenous variables jump to stable solution

$$\tilde{y}_t = 0 \qquad \text{and} \qquad \pi_t = 0$$

- Therefore

$$i_t = r_t^n$$

- But an *equilibrium outcome*, not a description of the *policy rule*

- "*Off-equilibrium threat*" of sufficient reaction

- Under these conditions, feedback rule implements optimal policy

  – other rules can also implement optimal policy

Next class

- Monetary policy in the new Keynesian model, part three
  - simple rules (Taylor rules, etc)
- Reading: Gali, chapter 4 section 4.4