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Advanced Macroeconomics
Tutorial #9: Solutions

Commitment vs. discretion in the new Keynesian model. Suppose the monetary authority seeks to minimize the expected discounted loss function

$$L = \mathbb{E}_0\left\{\sum_{t=0}^{\infty}\beta^t\frac{1}{2}\big(\hat{x}_t^2+\lambda\hat{\pi}_t^2\big)\right\}, \qquad \lambda > 0$$

subject to the modified new Keynesian Phillips curve

$$\hat{\pi}_t = \beta\mathbb{E}_t\{\hat{\pi}_{t+1}\} + \kappa\hat{x}_t + u_t$$

where the cost push shocks u_t follow a stationary AR(1) process

$$u_{t+1} = \rho_u u_t + \varepsilon_{t+1}, \qquad 0 \le \rho_u < 1$$

and where the innovations ε_t are IID normal with mean zero and variance σ_u^2. Suppose for simplicity that the natural real rate is constant and equal to the rate of time preference $r^n = \rho = 1/\beta - 1 > 0$.

To begin with, suppose the monetary authority cannot commit to future actions.

(a) Derive the monetary authority's optimal discretionary policy for inflation and the output gap and the dynamics of inflation implied by this discretionary policy.

(b) Guess that, in this scenario, inflation and the output gap are linear in the cost push shock, $\hat{\pi}_t = \psi_{\pi u}u_t$ and $\hat{x}_t = \psi_{xu}u_t$ for two coefficients $\psi_{\pi u}$ and ψ_{xu}. Use the method of undetermined coefficients to solve for $\psi_{\pi u}$ and ψ_{xu}. Explain intuitively how inflation and the output gap respond to a cost push shock. What does this imply for the equilibrium path of nominal interest rates?

Now suppose the monetary authority can commit to future actions.

(c) Derive the monetary authority's optimal policy with commitment and the dynamics of inflation implied by this policy.

(d) Guess that, in this scenario, the *price level* satisfies a law of motion of the form

$$\hat{p}_t = \psi_{pp}\hat{p}_{t-1} + \psi_{pu}u_t$$

for two coefficients ψ_{pp} and ψ_{pu}. Use the method of undetermined coefficients to solve for ψ_{pp} and ψ_{pu}. Explain intuitively how the price level, inflation, and the output gap respond to a cost push shock. What does this imply for the equilibrium path of nominal interest rates?

Now suppose the parameter values $\beta = 1/1.02$, $\kappa = 0.17$ and for the cost push shock process $\rho_u = 0.8$, $\sigma_u = 0.015$ and that the monetary authority's weight on inflation is $\lambda = 1$.

(e) Calculate the long-run variances of inflation, the output gap and nominal interest rate under both the discretion and commitment scenarios. Use these variances to calculate the (normalized) loss $2(1-\beta)L$ under these two scenarios. Give intuition for your findings.

(f) Suppose the economy is at steady state and that at $t = 0$ there is a 1 standard deviation cost push shock, i.e., $\varepsilon_0 = \sigma_u$. Calculate and plot for $T = 50$ periods after the shock the impulse response functions of inflation, the output gap, nominal interest rates and the price level under the two scenarios. Explain how these impulse response functions compare across the two scenarios.

(g) Now suppose the monetary authority gives more weight to inflation, $\lambda = 5$. How do your results for (e) and (f) change? What if $\lambda = 10$? What if $\lambda = 50$? Explain.

Solutions:

(a) If the monetary authority cannot commit to future actions then its problem is essentially static and reduces to minimizing the one-period loss

$$\frac{1}{2}\left(\hat{x}_t^2 + \lambda\hat{\pi}_t^2\right)$$

subject to

$$\hat{\pi}_t = \kappa\hat{x}_t + \xi_t, \qquad \xi_t \equiv \beta\mathbb{E}_t\left\{\hat{\pi}_{t+1}\right\} + u_t$$

where ξ_t denotes terms that the monetary authority takes as given. Substituting out $\hat{\pi}_t$ in the objective, the monetary authority chooses $\hat{x}_t$ to minimize

$$\frac{1}{2}\left(\hat{x}_t^2 + \lambda(\kappa\hat{x}_t + \xi_t)^2\right)$$

The first order condition for this problem can be written

$$\hat{x}_t = -\kappa\lambda\hat{\pi}_t$$

so that the monetary authority 'leans-against-the-wind' in the sense of reducing the output gap when inflation rises. Then plugging this into the new Keynesian Phillips curve gives

$$\hat{\pi}_t = \beta\mathbb{E}_t\left\{\hat{\pi}_{t+1}\right\} - \kappa^2\lambda\hat{\pi}_t + u_t$$

or

$$\hat{\pi}_t = \frac{\beta}{1+\kappa^2\lambda}\mathbb{E}_t\left\{\hat{\pi}_{t+1}\right\} + \frac{1}{1+\kappa^2\lambda}u_t$$

This is a single stochastic difference equation in $\hat{\pi}_t$ taking as given the process for the cost push shocks u_t.

(b) If $\hat{\pi}_t = \psi_{\pi u}u_t$ then $\mathbb{E}_t\left\{\hat{\pi}_{t+1}\right\} = \psi_{\pi u}\rho_u u_t$ so that on plugging these expressions into the stochastic difference equation for inflation we have

$$\psi_{\pi u}u_t = \frac{\beta}{1+\kappa^2\lambda}\psi_{\pi u}\rho_u u_t + \frac{1}{1+\kappa^2\lambda}u_t$$

Since this must hold for every u_t we have the restriction

$$\psi_{\pi u} = \frac{\beta}{1+\kappa^2\lambda}\psi_{\pi u}\rho_u + \frac{1}{1+\kappa^2\lambda}$$

and hence

$$\psi_{\pi u} = \frac{1}{(1-\beta\rho_u)+\kappa^2\lambda}$$

Since the discretionary policy is $\hat{x}_t = -\kappa\lambda\hat{\pi}_t$ we then have

$$\psi_{xu} = -\kappa\lambda\psi_{\pi u} = -\frac{\kappa\lambda}{(1-\beta\rho_u)+\kappa^2\lambda}$$

Hence a cost push shock $u_t > 0$ increases inflation $\hat{\pi}_t > 0$ and reduces the output gap $\hat{x}_t < 0$ (just like an adverse aggregate supply shock in a static AS-AD model).
The nominal interest rate is then given by the dynamic IS curve, which can be written

$$i_t = \rho + \sigma\mathbb{E}_t\left\{\Delta\hat{x}_{t+1}\right\} + \mathbb{E}_t\left\{\hat{\pi}_{t+1}\right\}$$

using the assumption that that natural real rate is a constant $r^n = \rho$. Given the solutions for the output gap and inflation we then have

$$\mathbb{E}_t\left\{\Delta\hat{x}_{t+1}\right\} = \psi_{xu}(\rho_u - 1)u_t = \frac{\kappa\lambda(1-\rho_u)}{(1-\beta\rho_u)+\kappa^2\lambda}u_t$$

and

$$\mathbb{E}_t\left\{\hat{\pi}_{t+1}\right\} = \psi_{\pi u}\rho_u u_t = \frac{\rho_u}{(1-\beta\rho_u)+\kappa^2\lambda}u_t$$

so that

$$i_t = \rho + \sigma\left[\frac{\kappa\lambda(1-\rho_u)}{(1-\beta\rho_u)+\kappa^2\lambda}\right]u_t + \left[\frac{\rho_u}{(1-\beta\rho_u)+\kappa^2\lambda}\right]u_t$$

or simply

$$i_t = \rho + \left[\frac{\sigma\kappa\lambda(1-\rho_u)+\rho_u}{(1-\beta\rho_u)+\kappa^2\lambda}\right]u_t$$

Hence in equilibrium a cost push shock $u_t > 0$ leads to a higher nominal interest rate $i_t > \rho$, both because the output gap is expected to rise (as the level of the output gap mean-reverts) and because expected inflation is positive.

(c) Under commitment, the monetary authority solves a genuine intertemporal problem. Its Lagrangian can be written

$$\mathcal{L} = \mathbb{E}_0\left\{\sum_{t=0}^{\infty}\beta^t\left[\frac{1}{2}\left(\hat{x}_t^2 + \lambda\hat{\pi}_t^2\right) + \mu_t\left(\hat{\pi}_t - \beta\hat{\pi}_{t+1} - \kappa\hat{x}_t - u_t\right)\right]\right\}$$

where $\mu_t \geq 0$ denotes the multipliers on the new Keynesian Phillips curve. The key first order conditions for this problem are, for the output gap

$$\hat{x}_t = \kappa\mu_t$$

and for inflation

$$\lambda\hat{\pi}_t = -\mu_t + \mu_{t-1} = -\Delta\mu_t$$

Differencing the optimality condition for the output gap gives

$$\Delta\hat{x}_t = \kappa\Delta\mu_t$$

Hence on eliminating the multipliers

$$\Delta\hat{x}_t = -\kappa\lambda\hat{\pi}_t$$

We again get a form of 'leaning-against-the-wind' but with the seemingly subtle difference that now when inflation is high $\hat{\pi}_t > 0$ the monetary authority seeks to reduce output gap *growth* $\Delta\hat{x}_t < 0$ rather than the level of the output gap as was the case in part (a) above. Note that the level of the output gap is related to the changes via

$$\hat{x}_t = \sum_{k=0}^{t} \Delta\hat{x}_k$$

and likewise the price level is related to inflation via

$$\hat{p}_t = \sum_{k=0}^{t} \hat{\pi}_k$$

Hence in terms of the levels, this policy is equivalent to

$$\hat{x}_t = -\kappa\lambda\hat{p}_t$$

To make use of this, write the new Keynesian Phillips curve in terms of the price level

$$\hat{p}_t - \hat{p}_{t-1} = \beta\mathbb{E}_t\left\{\hat{p}_{t+1} - \hat{p}_t\right\} + \kappa\hat{x}_t + u_t$$

Then use $\hat{x}_t = -\kappa\lambda\hat{p}_t$ to eliminate the output gap and solve for $\hat{p}_t$. This gives a single stochastic difference equation in the price level

$$\hat{p}_t = \gamma\hat{p}_{t-1} + \gamma\beta\mathbb{E}_t\left\{\hat{p}_{t+1}\right\} + \gamma u_t$$

where the coefficient γ is defined by

$$\gamma \equiv \frac{1}{1+\beta+\kappa^2\lambda} \in (0,1)$$

Solving this difference equation gives us the dynamics of the price level $\hat{p}_t$ from which we can recover inflation $\hat{\pi}_t = \hat{p}_t - \hat{p}_{t-1}$ and the output gap $\hat{x}_t = -\kappa\lambda\hat{p}_t$.

(d) If

$$\hat{p}_t = \psi_{pp}\hat{p}_{t-1} + \psi_{pu}u_t$$

then

$$\mathbb{E}_t\left\{\hat{p}_{t+1}\right\} = \psi_{pp}(\psi_{pp}\hat{p}_{t-1} + \psi_{pu}u_t) + \psi_{pu}\rho_u u_t$$

Plugging these expressions into the stochastic difference equation in the price level gives

$$\psi_{pp}\hat{p}_{t-1} + \psi_{pu}u_t = \gamma\hat{p}_{t-1} + \gamma\beta\left[\psi_{pp}(\psi_{pp}\hat{p}_{t-1} + \psi_{pu}u_t) + \psi_{pu}\rho_u u_t\right] + \gamma u_t$$

Collecting terms we have

$$0 = \left[\psi_{pp} - \gamma - \gamma\beta\psi_{pp}^2\right]\hat{p}_{t-1} + \left[\psi_{pu}(1 - \gamma\beta(\psi_{pp} + \rho_u) - \gamma\right]u_t$$

which implies the restrictions

$$\gamma\beta\psi_{pp}^2 - \psi_{pp} + \gamma = 0$$

and

$$\psi_{pu} = \frac{\gamma}{1 - \gamma\beta(\psi_{pp} + \rho_u)} = \frac{\psi_{pp}}{1 - \beta\rho_u\psi_{pp}} > \psi_{pp}$$

(where the latter equality follows using $\gamma\beta\psi_{pp} = 1 - \frac{\gamma}{\psi_{pp}}$). In short, we solve for roots of the quadratic $\gamma\beta\psi_{pp}^2 - \psi_{pp} + \gamma = 0$, set ψ_{pp} equal to the stable root and then recover ψ_{pu} from the last expression. Once we have these coefficients we have $\hat{p}_t$ from which we can recover inflation $\hat{\pi}_t = \hat{p}_t - \hat{p}_{t-1}$ and the output gap $\hat{x}_t = -\kappa\lambda\hat{p}_t$. Starting from steady-state, on impact a cost push shock $u_0 > 0$ increases the price level and hence increases inflation $\hat{\pi}_0 = \hat{p}_0 > 0$ in the short run which in turn implies that the output gap falls $\hat{x}_0 < 0$. Over time, the price level is brought back to its long run level which necessitates a period of negative inflation $\hat{\pi}_t < 0$ and hence increasing output gaps $\Delta\hat{x}_t > 0$ as output is brought back to its long-run level from below. These dynamics are easily seen in the impulse response functions in part (f) below.

We can again back out the equilibrium nominal interest rate from the dynamic IS curve

$$i_t = \rho + \sigma\mathbb{E}_t\left\{\Delta\hat{x}_{t+1}\right\} + \mathbb{E}_t\left\{\hat{\pi}_{t+1}\right\}$$

Where now we have

$$\mathbb{E}_t\left\{\Delta\hat{x}_{t+1}\right\} = -\kappa\lambda\mathbb{E}_t\left\{\hat{\pi}_{t+1}\right\}$$

Hence

$$i_t = \rho + (1 - \sigma\kappa\lambda)\mathbb{E}_t\left\{\hat{\pi}_{t+1}\right\}$$

(compare this to the expression in Lecture 17 slide 6). To evaluate this, observe that since

$$\hat{p}_{t+1} = \psi_{pp}\hat{p}_t + \psi_{pu}u_{t+1}$$

we also have

$$\hat{\pi}_{t+1} = \psi_{pp}\hat{\pi}_t + \psi_{pu}\Delta u_{t+1}$$

so that

$$\mathbb{E}_t\left\{\hat{\pi}_{t+1}\right\} = \psi_{pp}\hat{\pi}_t + \psi_{pu}(\rho_u - 1)u_t$$

Hence equilibrium interest rates are given by

$$i_t = \rho + (1 - \sigma\kappa\lambda)(\psi_{pp}\hat{\pi}_t + \psi_{pu}(\rho_u - 1)u_t)$$

In the special case $\sigma\kappa\lambda = 1$, nominal interest rates are constant independent of the cost push shock. More generally, the sign of the response is ambiguous and depends on whether $\sigma\kappa\lambda$ is more or less than 1.

(e) The loss function can be written

$$L = \mathbb{E}_0\left\{\sum_{t=0}^{\infty}\beta^t\frac{1}{2}\left(\hat{x}_t^2 + \lambda\hat{\pi}_t^2\right)\right\} = \frac{1}{2}\sum_{t=0}^{\infty}\beta^t\mathbb{E}_0\left\{\hat{x}_t^2 + \lambda\hat{\pi}_t^2\right\} = \frac{1}{2}\sum_{t=0}^{\infty}\beta^t\left(\mathbb{E}_0\left\{\hat{x}_t^2\right\} + \lambda\mathbb{E}_0\left\{\hat{\pi}_t^2\right\}\right)$$

and since $\hat{x}_t$ and $\hat{\pi}_t$ are both stationary and mean zero, $\mathbb{E}_0\left\{\hat{x}_t^2\right\} = \text{Var}\{\hat{x}_t\}$ and $\mathbb{E}_0\left\{\hat{\pi}_t^2\right\} = \text{Var}\{\hat{\pi}_t\}$ so that the loss is

$$L = \frac{1}{2}\sum_{t=0}^{\infty}\beta^t\left(\text{Var}\{\hat{x}_t\} + \lambda\text{Var}\{\hat{\pi}_t\}\right) = \frac{1}{2(1-\beta)}\left(\text{Var}\{\hat{x}_t\} + \lambda\text{Var}\{\hat{\pi}_t\}\right)$$

and the normalized loss $\hat{L} \equiv 2(1-\beta)L$ is therefore just

$$\hat{L} = \text{Var}\{\hat{x}_t\} + \lambda\text{Var}\{\hat{\pi}_t\}$$

The attached Dynare file `tutorial9.mod` solves the model with the given parameters under either discretion or commitment (just comment out line 24 or 25 depending on which case you want to solve). For the discretionary case we get $\text{Var}\{\hat{x}_t\} = 0.0003$ and $\text{Var}\{\hat{\pi}_t\} = 0.0104$ so that with $\lambda = 1$ the normalized loss is $\hat{L} = 0.0107$. For the commitment case we get $\text{Var}\{\hat{x}_t\} = 0.0023$ and $\text{Var}\{\hat{\pi}_t\} = 0.0028$ so that with $\lambda = 1$ the normalized loss is $\hat{L} = 0.0051$. Hence the welfare loss under commitment is about one-half of the welfare loss under discretion — i.e., as anticipated the outcome with commitment is better.[1]

(f) Figure 1 shows the impulse response functions under discretion. Notice that since inflation is positive throughout, the price level permanently increases and the output gap is negative throughout, returning to zero from below as inflation returns to zero from above. As we saw in part (b), under discretion the output gap and inflation are linear functions of the cost push shock and so inherit its AR(1) dynamics.

Figure 2 shows the impulse response functions under commitment. Notice that while inflation is positive in the short run there is ultimately a period of negative inflation so that the price level is brought back to its original level. The output gap is negative throughout. The *change* in the output gap, $\Delta\hat{x}_t$, is negative when inflation is positive and is positive when inflation is negative (as output is again brought back to zero from below). Hence unlike in the case of discretion, the output gap keeps falling while inflation is positive — reaching a trough of about -0.015 under commitment as opposed to the impact effect of about -0.01 under discretion — only rising when inflation switches sign. This gives the output gap a 'hump-shaped' that is mirrored in the behavior of the price level. These patterns are quite different from the AR(1) dynamics under discretion.

(g) Figures 3 and 4 show the impulse response functions for the case $\lambda = 5$ under discretion and commitment respectively. In both cases, the inflation response on impact is smaller and the output gap is more negative as the monetary authority more aggressively stabilizes inflation at the expense of output volatility. In the case of commitment, the persistence of inflation is also lower (in the case of discretion, the persistence of inflation is simply the exogenous persistence of the shock and so is unaffected by λ).

[1] You can check Dynare's calculations by noting that, under discretion, $\text{Var}\{\hat{\pi}_t\} = \psi_{\pi u}^2\sigma_u^2/(1-\rho_u^2)$ and similarly $\text{Var}\{\hat{x}_t\} = \psi_{xu}^2\sigma_u^2/(1-\rho_u^2)$ and using the formulas for $\psi_{\pi u}, \psi_{xu}$ given in part (b) evaluated at the given parameter values.

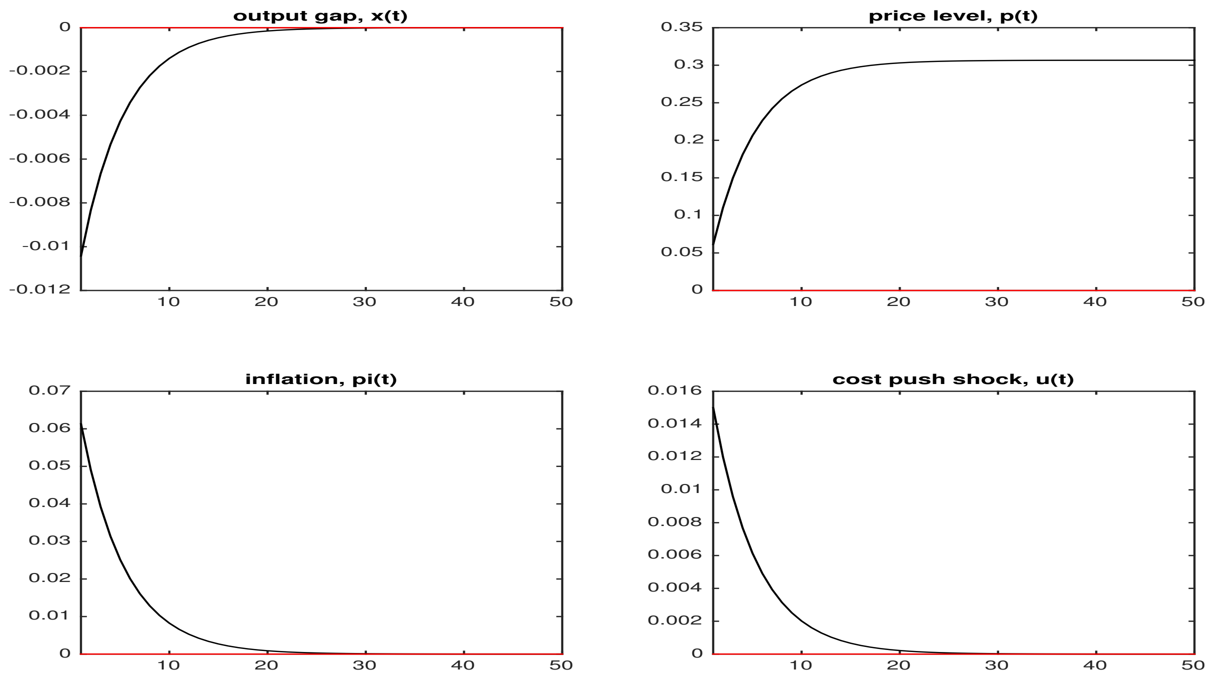

Figure 1: Discretion, $\lambda = 1$

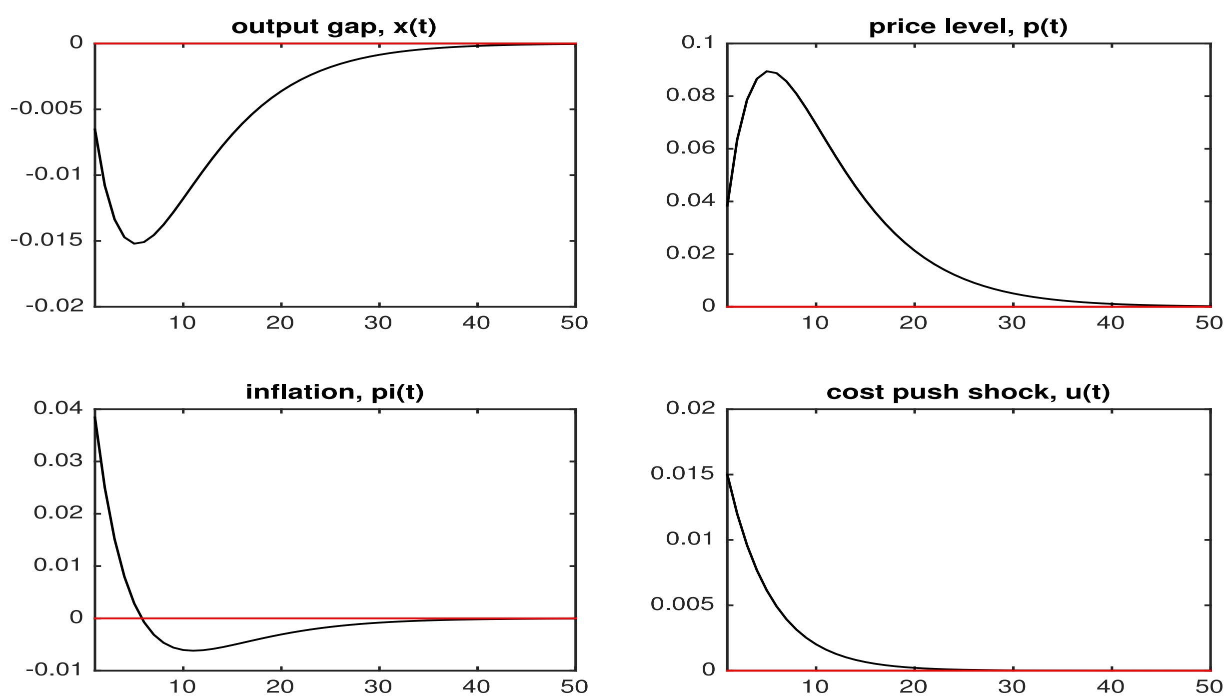

Figure 2: Commitment, $\lambda = 1$

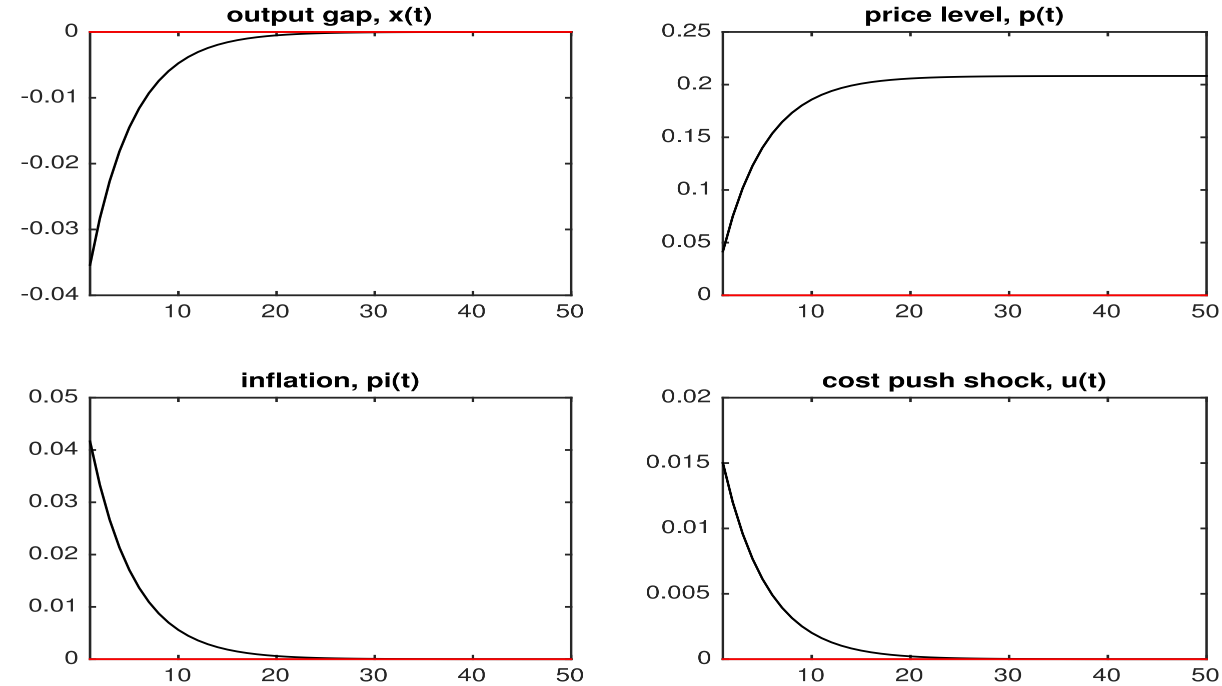

Figure 3: Discretion, $\lambda = 5$

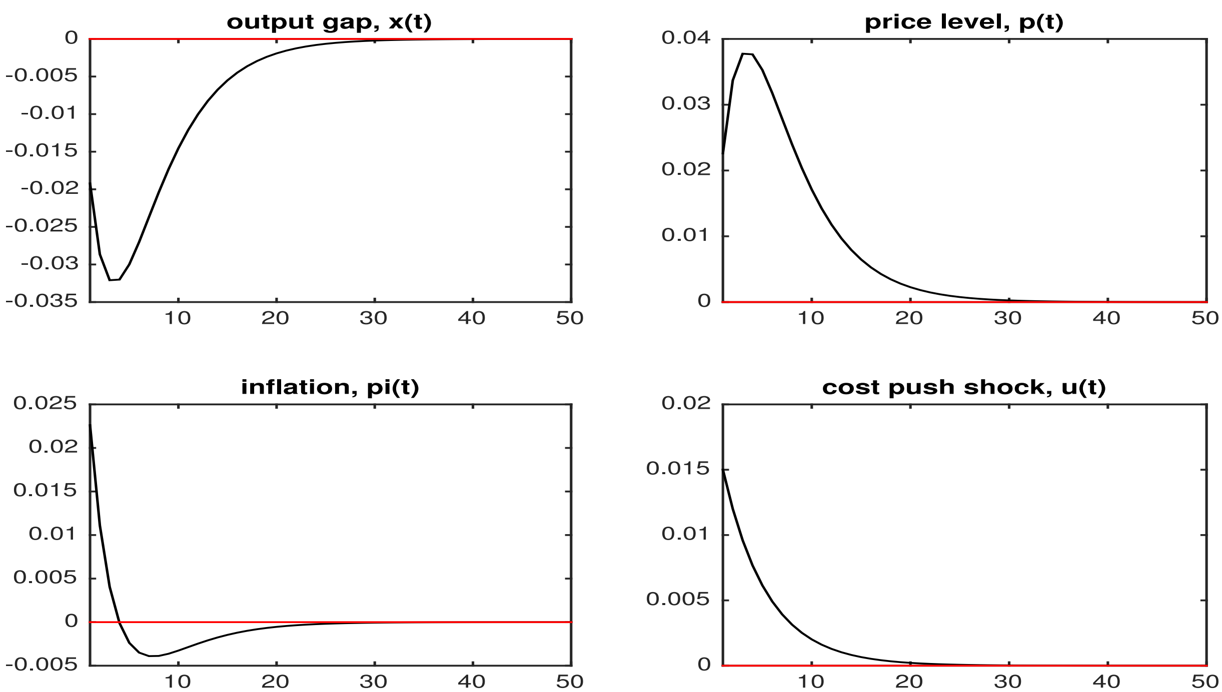

Figure 4: Commitment, $\lambda = 5$

The table below lists the variances of inflation and the output gap under discretion and commitment for each of $\lambda = 1, 5, 10, 50$. In each case, commitment involves smaller welfare losses than discretion. For each scenario, we see that as λ increases the variance of inflation falls while the variance of the output gap rises. Whether the normalized loss $\hat{L}$ is increasing or decreasing in λ depends on the parameter details. For example, increasing from $\lambda = 1$ to $\lambda = 5$ or $\lambda = 10$ increases the loss and so is welfare reducing (since the decrease in the variance of inflation is not enough to offset the increased weight on inflation volatility), while increasing further to $\lambda = 50$ brings the loss back down as the variance of inflation approaches zero so that the high weight on it doesn't matter.

	discretion			commitment		
λ	Var$\{\hat{x}_t\}$	Var$\{\hat{\pi}_t\}$	loss $\hat{L}$	Var$\{\hat{x}_t\}$	Var$\{\hat{\pi}_t\}$	loss $\hat{L}$
1	0.0003	0.0104	0.0107	0.0023	0.0028	0.0051
5	0.0035	0.0048	0.0275	0.0068	0.0008	0.0108
10	0.0071	0.0025	0.0321	0.0095	0.0004	0.0135
50	0.0164	0.0002	0.0264	0.0158	0.0000	0.0158