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ECON40002
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Advanced Macroeconomics
Tutorial #7: Solutions

Stochastic growth model in Dynare, nonstationary version. Consider a stochastic growth model with constant labor force $L > 0$ where output is given by the production function

$$Y_t = K_t^{\alpha}(A_t L)^{1-\alpha}, \qquad 0 < \alpha < 1$$

Let $g_t \equiv A_t/A_{t-1}$ denote the gross growth rate of labor-augmenting productivity and suppose that this growth rate follows a stationary AR(1) process in logs

$$\log g_{t+1} = (1-\phi)\log \bar{g} + \phi \log g_t + \varepsilon_{t+1}, \qquad 0 < \phi < 1, \qquad \bar{g} \geq 1$$

where the innovations ε_t are IID $N(0, \sigma_\varepsilon^2)$. For the sake of interpretation, note that $\log g_t = \Delta \log A_t$, i.e., is approximately the net growth rate. Now suppose that the planner maximizes

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \log(C_t/L)\, L, \qquad 0 < \beta < 1$$

where C_t/L denotes consumption per worker, subject to the resource constraints

$$C_t + K_{t+1} = K_t^{\alpha}(A_t L)^{1-\alpha} + (1-\delta)K_t, \qquad 0 < \delta < 1$$

given initial conditions $K_0, A_0 > 0$ and the process for productivity growth above.

(a) Let $c_t = C_t/A_t L$ and $k_{t+1} = K_{t+1}/A_t L$ etc denote the detrended levels of consumption, capital etc. Derive the planner's key optimality conditions for c_t and k_{t+1}.

(b) Solve for the non-stochastic steady state values of the detrended levels $\bar{c}$ and $\bar{k}$. What is the effect of an increase in the long-run average growth rate $\bar{g}$ on $\bar{c}$ and $\bar{k}$? Explain.

Now suppose the following parameter values: $\alpha = 0.3$, $\beta = 0.98$, $\delta = 0.02$ and for the growth process $\bar{g} = 1.02$, $\phi = 0$ and $\sigma_\varepsilon = 0.015$. Use Dynare to do the following:

(c) Calculate the long-run standard deviations of productivity growth and the detrended levels of consumption, capital, output and investment. Which of these variables move most closely together? Which of these variables is most volatile? Explain.

(d) Suppose the economy is at steady state and that at $t = 0$ there is a 1 standard deviation innovation to productivity growth, i.e., $\varepsilon_0 = \sigma_\varepsilon$. Calculate and plot for $T = 50$ periods after the shock the impulse response functions for both (i) the detrended levels of consumption, capital, output and investment, and (ii) the growth rates $\Delta \log C_t$, $\Delta \log K_t$, $\Delta \log Y_t$, $\Delta \log I_t$ and $\Delta \log A_t$. Give intuition for your findings.

(e) Now let $K_0 = A_0 = 1$ and simulate a sequence of productivity growth $\log g_t$ of length $T = 250$ and use this to generate simulated sequences of $\log C_t$, $\log K_t$, $\log Y_t$ and $\log A_t$. Explain the dynamics of these variables and the dynamics of the ratios $\log(C_t/Y_t)$, and $\log(K_t/Y_t)$. Give intuition for your findings.

(f) Suppose instead that $\phi = 0.5$. How if at all do your answers to (c), (d) and (e) change? Explain.

Solutions

(a) Start with the resource constraint in levels

$$C_t + K_{t+1} = K_t^\alpha (A_t L)^{1-\alpha} + (1-\delta)K_t, \qquad 0 < \delta < 1$$

Divide both sides by $A_t L$ to get

$$\frac{C_t}{A_t L} + \frac{K_{t+1}}{A_t L} = \frac{K_t^\alpha (A_t L)^{1-\alpha}}{A_t L} + (1-\delta)\frac{K_t}{A_t L}$$

or

$$\frac{C_t}{A_t L} + \frac{K_{t+1}}{A_t L} = \left(\frac{K_t}{A_t L}\right)^\alpha + (1-\delta)\frac{K_t}{A_t L}$$

Now let $c_t = C_t/A_t L$ and note carefully that $k_{t+1} = K_{t+1}/A_t L$ so that $k_t = K_t/A_{t-1}L$ which implies $k_t = (K_t/A_t L)g_t$ given $g_t = A_t/A_{t-1}$. Making these substitutions, the resource constraint is, in detrended variables

$$c_t + k_{t+1} = \left(\frac{k_t}{g_t}\right)^\alpha + (1-\delta)\frac{k_t}{g_t}$$

so that the planner's Lagrangian can be written

$$\mathcal{L} = \mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t \log c_t + \sum_{t=0}^{\infty} \lambda_t \left[\left(\frac{k_t}{g_t}\right)^\alpha + (1-\delta)\frac{k_t}{g_t} - c_t - k_{t+1} \right] \right\}$$

The key first order conditions are, for detrended consumption

$$c_t : \qquad \beta^t \frac{1}{c_t} = \lambda_t$$

and for detrended capital

$$k_{t+1} : \qquad \lambda_t = \mathbb{E}_t \left\{ \lambda_{t+1} \left[\alpha k_{t+1}^{\alpha-1} g_{t+1}^{-\alpha} + (1-\delta) g_{t+1}^{-1} \right] \right\}$$

where the term in square brackets on the left is the gross return on capital in detrended units. Eliminating the Lagrange multipliers as usual gives the consumption Euler equation

$$\frac{1}{c_t} = \beta \mathbb{E}_t \left\{ \frac{1}{c_{t+1}} \left[\alpha k_{t+1}^{\alpha-1} g_{t+1}^{-\alpha} + (1-\delta) g_{t+1}^{-1} \right] \right\}$$

The resource constraint and consumption Euler equation plus initial k_0 and the transversality condition pin down the solution to the planner's problem in the usual way.

(b) In non-stochastic steady state we have the consumption Euler equation

$$1 = \beta \left(\alpha \bar{k}^{\alpha-1} \bar{g}^{-\alpha} + (1-\delta)\bar{g}^{-1} \right)$$

which determines $\bar{k}$ in terms of the long run growth rate $\bar{g}$, in particular

$$\bar{k} = \left(\frac{\alpha\beta}{\bar{g} - (1-\delta)\beta} \right)^{\frac{1}{1-\alpha}} \bar{g}$$

and we can then determine $\bar{c}$ from the resource constraint

$$\bar{c} = \left(\frac{\bar{k}}{\bar{g}} \right)^{\alpha} + (1-\delta)\frac{\bar{k}}{\bar{g}} - \bar{k}$$

(these collapse to the usual expressions if $\bar{g} = 1$).
The levels $\bar{k}$ and $\bar{c}$ are both *decreasing* in the long run growth rate $\bar{g}$. An increase in $\bar{g}$ increases the effective depreciation rate and reduces the amount of output that can be produced per effective worker, thereby reducing steady state capital per effective worker and steady state consumption per effective worker.

The attached Dynare file `tutorial7.mod` solves the model with the given parameters.

(c) This gives the key statistics

VARIABLE	MEAN	STD. DEV.	VARIANCE
c	0.4623	0.0229	0.0005
k	2.2951	0.0384	0.0015
g	0.0200	0.0150	0.0002

and the correlation matrix

Variables	c	k	g
c	1.0000	1.0000	-0.3641
k	1.0000	1.0000	-0.3641
g	-0.3641	-0.3641	1.0000

When $\phi = 0$ the productivity growth process is IID (log productivity is a random walk with drift). In this case, detrended capital and consumption are perfectly correlated (they are both positive functions of a single state variable $\hat{k}_t - \hat{g}_t$, i.e., in the log-linear solution the coefficients are $\psi_{kk} = -\psi_{kg}$ and $\psi_{ck} = -\psi_{cg}$), both are negatively correlated with productivity growth because of the way this increases 'effective depreciation'. Both detrended consumption and capital are more volatile than productivity growth with consumption smoother than capital.

(d) Figure 1 shows the results for the detrended levels, Figure 2 shows the results for the growth rates. As anticipated, on impact the detrended levels $\hat{c}_t$ and $\hat{k}_t$ both fall in response to $\hat{g}_t > 0$ with both then returning to zero from below as they return to steady state (equivalently, as the economy approaches its new long-run levels corresponding to a permanently higher productivity level). Notice that investment growth is more volatile than output growth which is more volatile than consumption growth.

(e) Figure 3 shows the results for the log levels, Figure 4 shows the results for $\log(C_t/Y_t)$ and $\log(K_t/Y_t)$. The log levels of consumption, output etc are the sum of a nonstationary process, $\log A_t$ and the stationary detrended levels, hence the log levels of consumption, output etc are likewise nonstationary. You can see here that the swings in investment are again larger than those in output and consumption. The (log) consumption/output ratio and capital/output ratio are stationary — i.e., consumption and output are *cointegrated* — but are quite persistent.

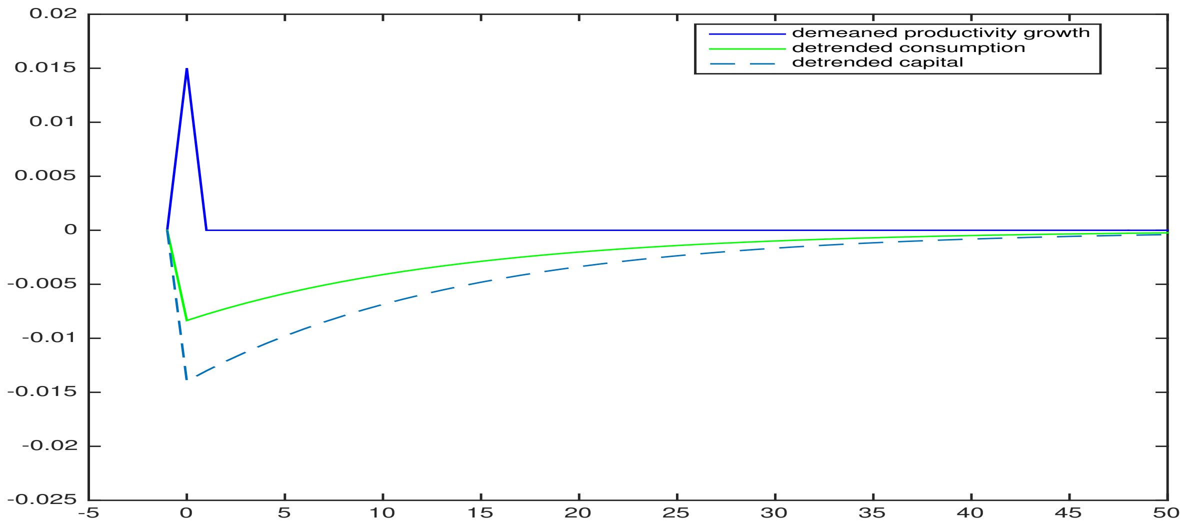

Figure 1: detrended levels when productivity growth g_t is IID

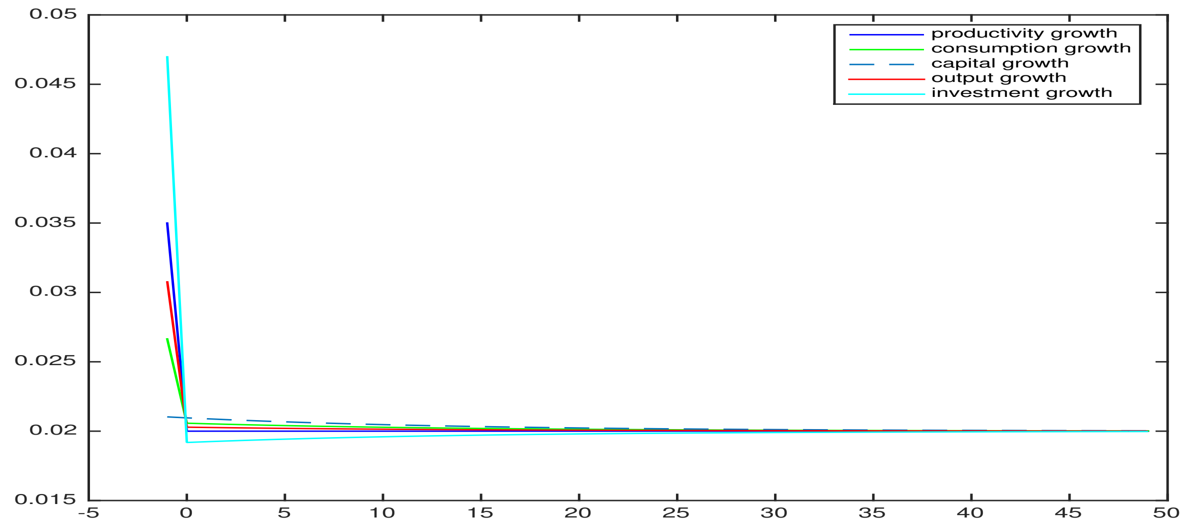

Figure 2: growth rates when productivity growth g_t is IID

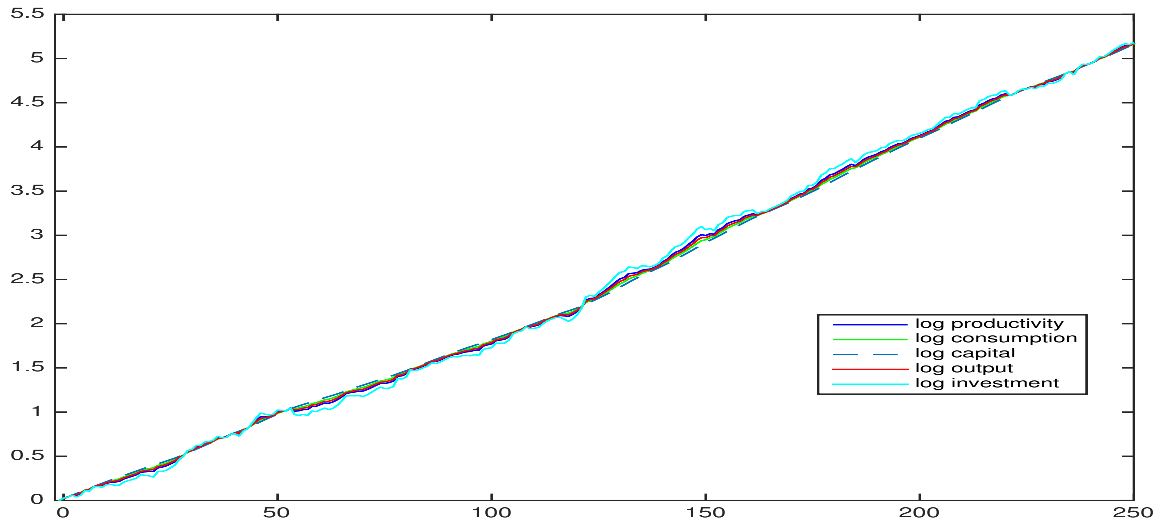

Figure 3: log levels when productivity growth g_t is IID

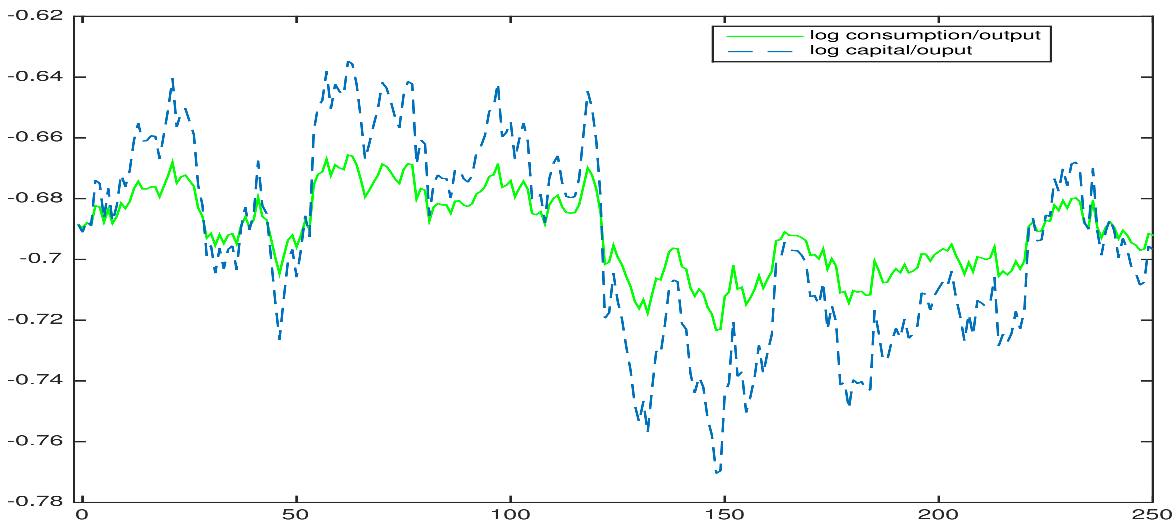

Figure 4: log 'great ratios' when productivity growth g_t is IID

(f) With $\phi = 0.5$, productivity growth is AR(1) and hence there is time-variation in expected productivity growth. If current productivity growth g_t is above long-run growth $\bar{g}$ then next period's productivity growth is also expected to be above long-run growth. For example, if $g_t = 1.04$ then productivity growth is expected to be $(0.5)(1.04) + (0.5)(1.02) = 1.03$ or 3% whereas with $\phi = 0$ productivity growth is always expected to be $\bar{g}$ regardless of the current realization.

We now get the key statistics

VARIABLE	MEAN	STD. DEV.	VARIANCE
c	0.4623	0.0442	0.0019
k	2.2951	0.0776	0.0060
g	0.0200	0.0173	0.0003

and the correlation matrix

Variables	c	k	g
c	1.0000	0.9912	-0.2873
k	0.9912	1.0000	-0.4115
g	-0.2873	-0.4115	1.0000

Now that productivity growth is serially correlated, detrended capital and consumption are no longer purely functions of a single state variable $\hat{k}_t - \hat{g}_t$, they now depend on each separately. Nonetheless, detrended capital and consumption are nearly perfectly correlated and both remain negatively correlated with productivity growth. Both remain more volatile than productivity growth with consumption smoother than capital.

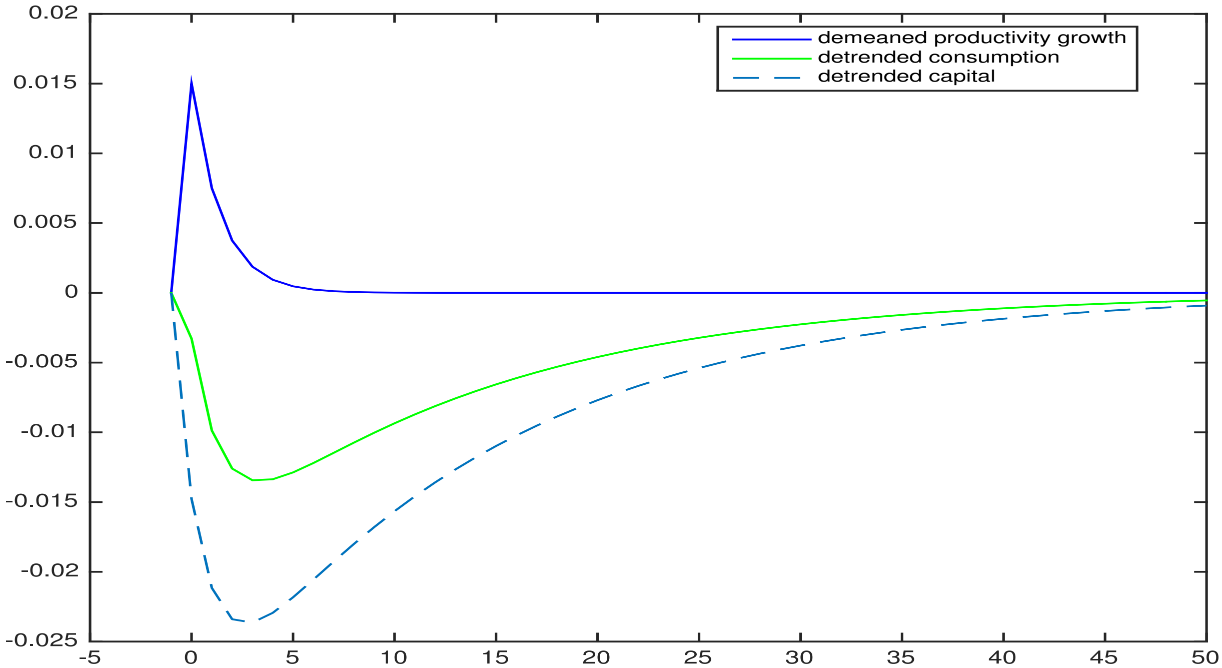

Figure 5: detrended levels when productivity growth g_t is AR(1)

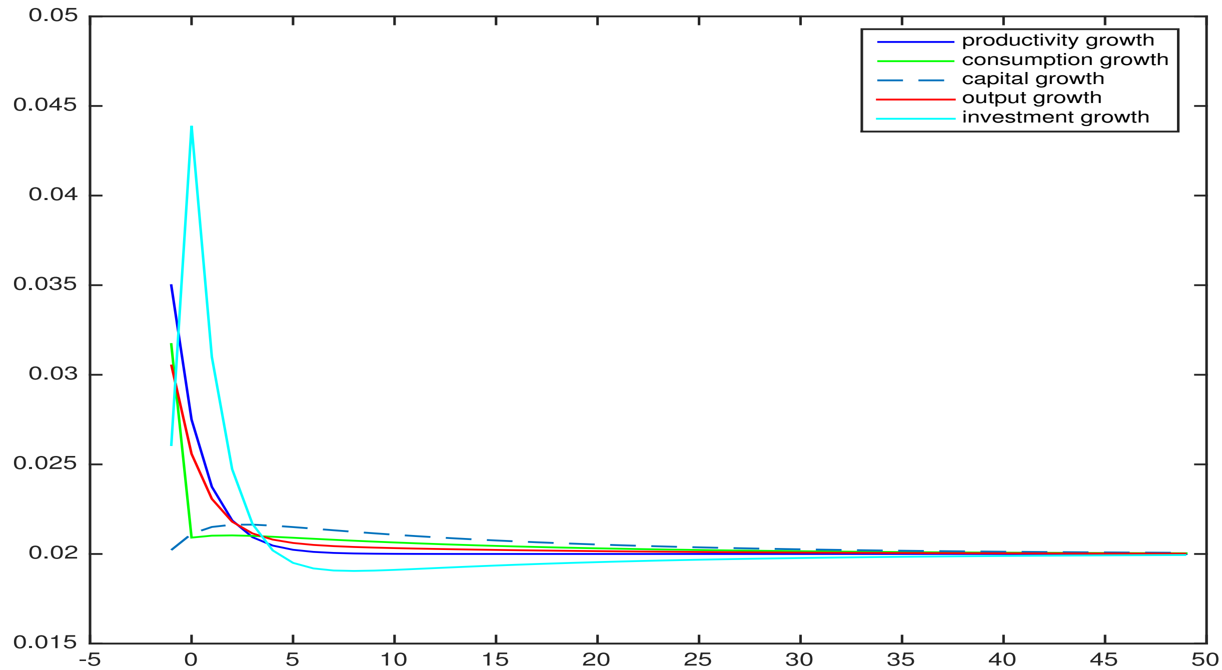

Figure 6: growth rates when productivity growth g_t is AR(1)

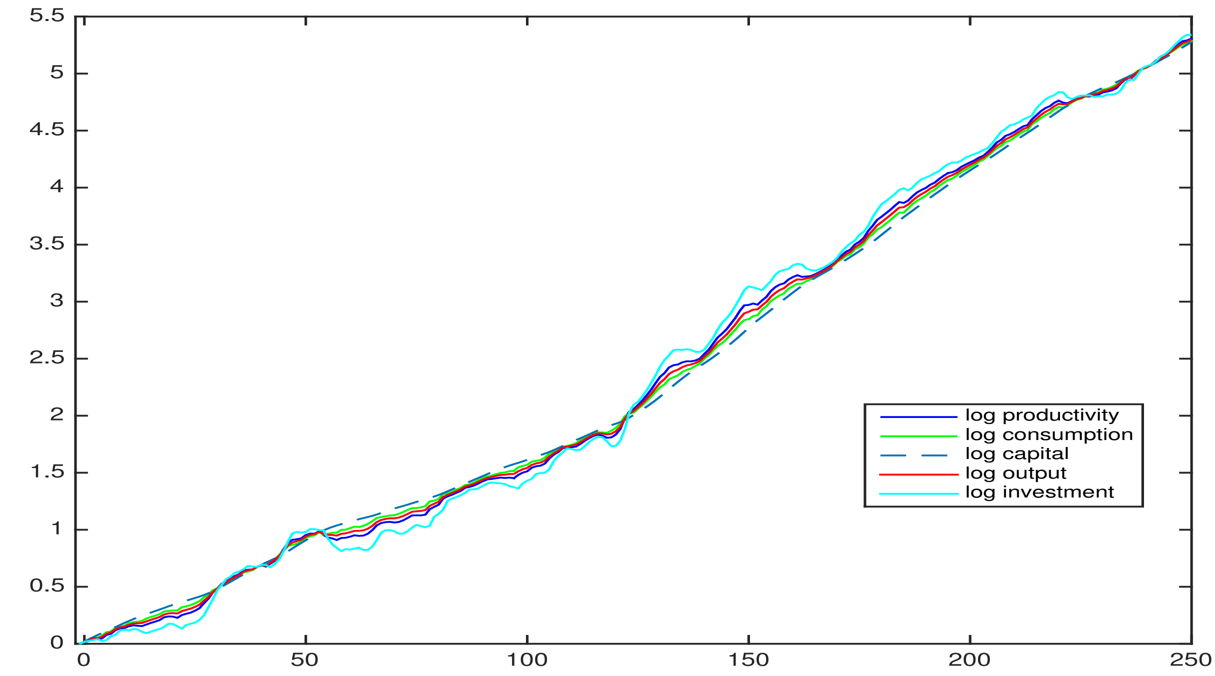

Figure 7: log levels when productivity growth g_t is AR(1)

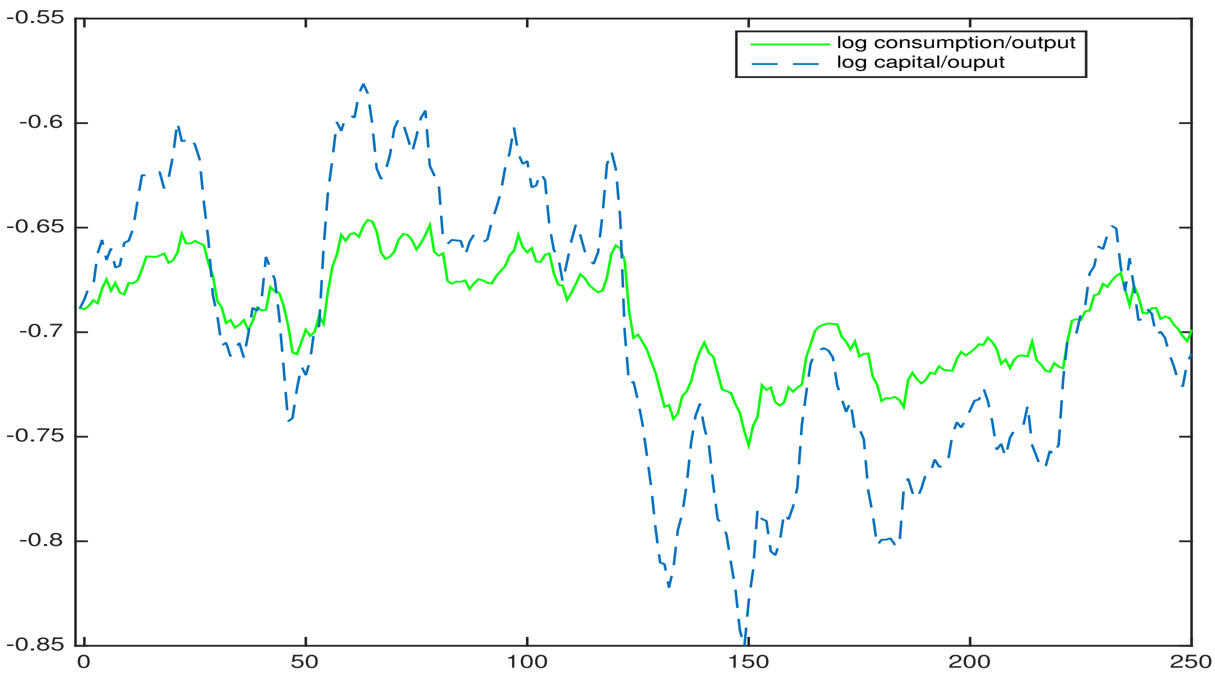

Figure 8: log 'great ratios' when productivity growth g_t is AR(1)