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ECON40002
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Advanced Macroeconomics
Tutorial #4: Solutions

Linear growth. Consider a continuous time Ramsey-Cass-Koopmans model. The planner seeks to maximize the intertemporal utility function

$$\int_0^\infty e^{-\rho t} \log(c(t))\, dt, \qquad \rho > 0$$

Output per person is given by the linear production function $y = Ak$ so that the planner's flow resource constraint is

$$\dot{k}(t) = (A - \delta)k(t) - c(t)$$

given initial $k(0) > 0$. Assume that the marginal product of capital is greater than the depreciation rate, $A > \delta$.

(a) Setup a Hamiltonian for this problem. Derive the key optimality conditions for $c(t)$ and $k(t)$.

(b) Let $r \equiv A - \delta$ and suppose $r > \rho$. Use a phase diagram to explain the dynamics of $c(t)$ and $k(t)$. Do these dynamics generally converge to a steady state c^* and k^*? Why or why not? Explain. How if at all would your answer change if instead $r < \rho$?

(c) Solve for the optimal $c(t)$ and $k(t)$ starting from initial conditions $c(0)$ and $k(0)$. What value must $c(0)$ have if these trajectories are to be optimal? Show that your solution satisfies the transversality condition.

(d) Show that the dynamics of $c(t)$ and $k(t)$ can be written as a linear system of the form

$$\begin{pmatrix} \dot{c}(t) \\ \dot{k}(t) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} c(t) \\ k(t) \end{pmatrix}$$

Express the coefficients a_{11} etc in terms of model parameters. Do the eigenvalues of this coefficient matrix imply stable or unstable dynamics? Explain.

Solutions:

(a) The Hamiltonian for this problem can be written

$$\mathcal{H} = u(c) + \mu(Ak - \delta k - c)$$

where μ denotes the multiplier on the resource constraint. The general form of optimality conditions for this kind of control problem are that the derivative of the Hamiltonian with respect to the control variable is zero, so here

$$\frac{\partial \mathcal{H}}{\partial c} = 0$$

and that the derivative of the Hamiltonian with respect to the state variable satisfies

$$\frac{\partial \mathcal{H}}{\partial k} = \rho\mu(t) - \dot{\mu}(t)$$

Computing the required derivatives

$$\frac{\partial \mathcal{H}}{\partial c} = u'(c) - \mu$$

and

$$\frac{\partial \mathcal{H}}{\partial k} = \mu(A - \delta)$$

So we can write these optimality conditions as

$$u'(c(t)) = \mu(t), \qquad \text{and} \qquad \dot{\mu}(t) = (\rho + \delta - A)\mu(t)$$

Using $u''(c(t))\dot{c}(t) = \dot{\mu}(t)$ we can combine these to get the consumption Euler equation

$$\frac{\dot{c}(t)}{c(t)} = \frac{A - \delta - \rho}{\sigma(c(t))}, \qquad \sigma(c) \equiv -\frac{u''(c)c}{u'(c)}$$

Now specifically we have $u(c) = \log c$ so that $\sigma(c) = 1$ meaning that for the problem at hand this simplifies to just

$$\frac{\dot{c}(t)}{c(t)} = A - \delta - \rho$$

We also have the resource constraint

$$\dot{k}(t) = (A - \delta)k(t) - c(t)$$

These are two differential equations in $c(t)$ and $k(t)$. To pin down the dynamics exactly we also need two boundary conditions. One boundary condition is the given initial capital $k(0) > 0$. The other boundary condition is the transversality condition, which can be written

$$\lim_{T \to \infty} e^{-\rho T} u'(c(T))k(T) = 0$$

where $u'(c) = 1/c$.

(b) Let $r \equiv A - \delta$ so that the consumption Euler equation is just

$$\frac{\dot{c}(t)}{c(t)} = r - \rho$$

Notice that the RHS is a constant, $r - \rho$, and does not depend on $k(t)$. With the linear production function there is no diminishing returns to capital so the marginal product of capital does not depend on $k(t)$.

In terms of a phase diagram, the $c(t)$ flows always point the same way, with $c(t)$ always rising if $r > \rho$ and always falling if $r < \rho$. From the resource constraint we have

$$\dot{k}(t) > 0 \qquad \Leftrightarrow \qquad rk(t) > c(t)$$

In other words, the $\dot{k}(t) = 0$ isocline $c = rk$ is a straight line through the origin with slope $r > 0$. The $\dot{k}(t) = 0$ isocline divides the (k, c) plane into two regions, one below the curve where consumption is sufficiently low $c(t) < rk(t)$ so that the capital stock is increasing, and the other above the curve where consumption is sufficiently high $c(t) > rk(t)$ so that the capital stock is decreasing.

If $r > \rho$ then consumption grows forever and hence does not converge to a steady state c^*. If instead $r < \rho$ then consumption falls forever and asymptotically approaches the steady state $c^* = k^* = 0$. Graphically, steady-states are found where the $\dot{c}(t) = 0$ isocline intersects the $\dot{k}(t) = 0$ isocline. The latter is a straight line through the origin with slope $r > 0$. If $r \neq \rho$, the former is just the origin and so the only possible steady state is at the origin and moreover this can only be approached if $r < \rho$.

(c) We will solve for these dynamics by recasting the problem as a second order differential equation in $k(t)$ and then using the method of undetermined coefficients. The resource constraint says

$$\dot{k}(t) = rk(t) - c(t)$$

Differentiating with respect to t then gives

$$\ddot{k}(t) = r\dot{k}(t) - \dot{c}(t)$$

But since $\dot{c}(t) = (r - \rho)c(t)$ and $c(t) = rk(t) - \dot{k}(t)$ this is

$$\ddot{k}(t) = r\dot{k}(t) - (r - \rho)(rk(t) - \dot{k}(t))$$

Collecting terms we can write this as

$$\ddot{k}(t) - (r + g)\dot{k}(t) + grk(t) = 0$$

where $g \equiv r - \rho$ is the growth rate of consumption (which may be positive or negative). Now guess that $\dot{k}(t) = \lambda k(t)$, i.e., that $k(t) = e^{\lambda t}k(0)$, for some coefficient λ to be determined. This guess implies $\ddot{k}(t) = \lambda\dot{k}(t) = \lambda^2 k(t)$ so for this guess to work it would have to be the case that

$$\left[\lambda^2 - (r + g)\lambda + gr\right]k(t) = 0$$

for any $k(t) > 0$. But then this can only work if the term in square brackets is zero, i.e., if λ is a root of the characteristic polynomial

$$p(\lambda) = \lambda^2 - (r + g)\lambda + gr = 0$$

Applying the quadratic formula, the roots are

$$\lambda_1\,,\ \lambda_2 = \frac{(r + g) \pm \sqrt{(r + g)^2 - 4gr}}{2}$$

$$= \frac{(r + g) \pm \sqrt{(r - g)^2}}{2}$$

$$= r\,,\ g$$

In short, the roots are $\lambda_1 = r$ and $\lambda_2 = g$. Intuitively it seems that we will want the root $\lambda = g$ so that capital and consumption grow at the same rate. To see the problem with $\lambda = r$, recall that the resource constraint says

$$\dot{k}(t) = rk(t) - c(t)$$

But then if we choose $\lambda = r$ we have $\dot{k}(t) = rk(t)$ and hence $c(t) = 0$ and this cannot be optimal. Any feasible choice of consumption does better than this, so this cannot be the consumption path that the solves the planner's problem (not surprisingly, it also violates the transversality condition). By contrast, if we choose $\lambda = g$ we have $\dot{k}(t) = gk(t)$ so that from the resource constraint we have

$$c(t) = rk(t) - \dot{k}(t) = (r - g)k(t) = \rho k(t)$$

So indeed this solution implies that consumption and capital are proportional to each other and grow (or shrink) at the common rate g. To get this path, we then need to set $c(0) = \rho k(0)$.

To summarize, the optimal trajectories for $c(t), k(t)$ are, starting from the initial $k(0)$,

$$c(t) = e^{gt}\rho k(0), \qquad k(t) = e^{gt}k(0), \qquad g = r - \rho$$

These trajectories satisfy the transversality condition

$$\lim_{T\to\infty} e^{-\rho T}u'(c(T))k(T) = \lim_{T\to\infty} e^{-\rho T}\frac{k(T)}{c(T)} = \lim_{T\to\infty} e^{-\rho T}\frac{e^{gT}k(0)}{e^{gT}\rho k(0)} = \lim_{T\to\infty} e^{-\rho T}\frac{1}{\rho} = 0$$

Notice that the consumption function $c = \rho k$ that pins down the initial $c(0)$ simply says to set flow consumption such that the 'present-value' of consumption $c(t)/\rho$ is equal to the 'stock of wealth'. This is a simple version of the permanent income hypothesis. Graphically, if the consumption function $c = \rho k$ is flatter than the $\dot{k}(t) = 0$ isocline $c = rk$ (i.e., if $r > \rho$), then the initial consumption $c(0) = \rho k(0) < rk(0)$ is below the $\dot{k}(t) = 0$ isocline and the capital stock grows forever (with consumption growing forever) because the net marginal product of capital $r = A - \delta$ is higher than the discount rate ρ. But if the consumption function $c = \rho k$ is steeper than the $\dot{k}(t) = 0$ isocline $c = rk$ (i.e., if $r < \rho$), then the initial consumption $c(0) = \rho k(0) > rk(0)$ is above the $\dot{k}(t) = 0$ isocline and the capital stock shrinks towards $k^* = 0$ with consumption likewise shrinking towards $c^* = 0$ because the net marginal product of capital $r = A - \delta$ is lower than the discount rate ρ.

(d) We have the differential equations $\dot{c}(t) = gc(t)$ and $\dot{k}(t) = rk(t) - c(t)$ which can be written as the 2×2 system

$$\begin{pmatrix} \dot{c}(t) \\ \dot{k}(t) \end{pmatrix} = \begin{pmatrix} g & 0 \\ -1 & r \end{pmatrix} \begin{pmatrix} c(t) \\ k(t) \end{pmatrix}$$

Notice that the determinant of this matrix is $a_{11}a_{22} - a_{12}a_{21} = gr$ and the trace is $a_{11} + a_{12} = g + r$ and indeed the two roots of this system are g and r, as calculated in part (c) above. So this system definitely has one unstable root, $\lambda_1 = r > 0$. If $g = r - \rho < 0$ it has one stable root but if $g = r - \rho > 0$ it has two unstable roots. So if $g < 0$ this system is a saddle and the only stable trajectory is that associated with $g < 0$ and $c(t) \to 0$ and $k(t) \to 0$ as $t \to \infty$. If $g > 0$ then the system is unambiguously unstable and we saw in part (c) that the solution is for $c(t) \to \infty$ and $k(t) \to \infty$ as $t \to \infty$.

The linear production function $y = Ak$ does not satisfy the Inada conditions. If parameters are conducive to saving, in particular, if the net marginal product of capital is greater than the rate of time preference, $A - \delta > \rho$, then this allows for unbounded capital accumulation and consumption without running into diminishing returns. But likewise if parameters are not conducive to growth, in particular, if the net marginal product of capital is lower than the rate of time preference, $A - \delta < \rho$, there is nothing to prevent the economy collapsing completely. That is, with the linear production function $y = Ak$ there is no 'stabilizing' tendency of the marginal product of capital to increase as the capital stock falls.