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Advanced Macroeconomics
Tutorial #3: Solutions

Government consumption — a simple case. Suppose the planner seeks to maximize the intertemporal utility function

$$\sum_{t=0}^{\infty} \beta^t u(c_t, g_t), \qquad 0 < \beta < 1$$

subject to the sequence of resource constraints

$$c_t + g_t + k_{t+1} = F(k_t, A) + (1-\delta)k_t, \qquad 0 < \delta < 1$$

given initial $k_0 > 0$. Here g_t denotes government purchases that provide utility (think of this as public services). The period utility function $u(c, g)$ has positive but diminishing marginal utility for each good. All variables are in per worker units.

(a) Derive optimality conditions that characterize the solution to the planner's problem. Give intuition for those optimality conditions. Explain how these optimality conditions pin down the dynamics of c_t, g_t and k_t.

(b) Derive expressions characterizing steady state c^*, g^*, k^*, y^* in this economy. Do these steady state values depend on the period utility function? Explain.

Now suppose that the production function is Cobb-Douglas, $y = F(k, A) = k^\alpha A^{1-\alpha}$ with $0 < \alpha < 1$ and that the utility function is $u(c, g) = (1-\gamma)\log(c) + \gamma \log(g)$ with $0 < \gamma < 1$.

(c) Solve for steady state values c^*, g^*, k^*, y^* and for the shares c^*/y^* and g^*/y^* in terms of the parameters. How do these depend on γ? How do these depend on A? Explain. Suppose the specific values: $\alpha = 0.3$, $\beta = 0.99$, $\gamma = 0.3$, $\delta = 0.02$ and $A = 1$. Calculate c^*, g^*, k^*, y^*.

(d) Log-linearize the optimality conditions from (a) around the steady-state. Guess that in log-deviations

$$\hat{c}_t = \psi_{ck}\hat{k}_t$$

$$\hat{g}_t = \psi_{gk}\hat{k}_t$$

and

$$\hat{k}_{t+1} = \psi_{kk}\hat{k}_t$$

Use the method of undetermined coefficients and the parameter values from (c) to calculate $\psi_{ck}, \psi_{gk}, \psi_{kk}$. How if at all do these differ from the answers you would get if there was no government consumption, $\gamma = 0$? Explain.

(e) Suppose the economy is at steady state then suddenly at $t = 0$ there is a 1% permanent increase in the level of productivity from $A = 1$ to $A' = 1.01$. Explain *qualitatively* the transitional dynamics of the economy as it adjusts to its new long run values. What happens to the ratio of private to public consumption c_t/g_t?

SOLUTIONS:

(a) Setting up the Lagrangian

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(c_t, g_t) + \sum_{t=0}^{\infty} \lambda_t [F(k_t, A) + (1-\delta)k_t - c_t - g_t - k_{t+1}]$$

The key first order conditions for this problem are, for consumption,

$$c_t : \qquad \beta^t u_{c,t} = \lambda_t$$

for government purchases,

$$g_t : \qquad \beta^t u_{g,t} = \lambda_t$$

for capital,

$$k_{t+1} : \qquad \lambda_t = \lambda_{t+1}\big[F_{k,t+1} + 1 - \delta\big]$$

and for the multipliers,

$$\lambda_t : \qquad c_t + g_t + k_{t+1} = F(k_t, A) + (1-\delta)k_t$$

We also have the transversality condition

$$\lim_{T\to\infty} \beta^T u_{c,T} k_{T+1} = 0$$

In these expressions $u_{c,t}$ and $u_{g,t}$ denote the marginal utilities of consumption and government purchases and $F_{k,t}$ denotes the marginal product of capital. Eliminating the multipliers, we can reduce this to a standard consumption Euler equation

$$u_{c,t} = \beta u_{c,t+1}\big[F_{k,t+1} + 1 - \delta\big]$$

and resource constraint

$$c_t + g_t + k_{t+1} = F(k_t, A) + (1-\delta)k_t$$

and a single new static condition equating the marginal utilities of consumption and government purchases

$$u_{c,t} = u_{g,t}$$

(i.e., at an optimum the marginal rate of substitution between c_t and g_t equals the marginal rate of transformation, which here is just 1). We can solve the latter for optimal government purchases as an implicit function of consumption, $g_t = g(c_t)$ say, then use this to eliminate g_t from the consumption Euler equation[1] and the resource constraint giving us a system of difference equations in c_t, k_t as usual. We then have k_0 and the transversality condition to pin the time paths of c_t, k_t down from which we can recover the time path of $g_t = g(c_t)$.

[1] The marginal utility $u_{c,t}$ depends on both c_t and g_t unless $u(c_t, g_t)$ is separable in c_t and g_t.

(b) In steady state, we have as usual

$$1 = \beta\big[F_k(k^*, A) + 1 - \delta\big]$$

which pins down k^* and hence $y^* = F(k^*, A)$ independent of $u(c, g)$. We then have the pair of equations

$$u_c(c^*, g^*) = u_g(c^*, g^*)$$

and

$$c^* + g^* = F(k^*, A) - \delta k^*$$

While the sum $c^* + g^*$ is independent of $u(c, g)$, we need to know $u(c, g)$ to split the sum into its separate parts c^* and g^*. Hence c^* and g^* generally depend on the period utility function while k^* and y^* do not.

(c) With this utility function and production function our consumption Euler equation becomes

$$\frac{1-\gamma}{c_t} = \beta\frac{1-\gamma}{c_{t+1}}\left(\alpha\left(\frac{k_{t+1}}{A}\right)^{\alpha-1} + 1 - \delta\right)$$

which simplifies to

$$\frac{c_{t+1}}{c_t} = \beta\left(\alpha\left(\frac{k_{t+1}}{A}\right)^{\alpha-1} + 1 - \delta\right)$$

while the static optimality condition for c_t and g_t becomes

$$\frac{1-\gamma}{c_t} = \frac{\gamma}{g_t}$$

hence government purchases are proportional to consumption

$$g_t = \frac{\gamma}{1-\gamma}c_t$$

Thus in steady state

$$k^* = \left(\frac{\alpha}{\rho+\delta}\right)^{\frac{1}{1-\alpha}} A$$

where $\rho = 1/\beta - 1$ is the rate of time preference. We then have

$$y^* = \left(\frac{\alpha}{\rho+\delta}\right)^{\frac{\alpha}{1-\alpha}} A$$

and hence as usual

$$\frac{k^*}{y^*} = \frac{\alpha}{\rho+\delta}$$

Now write the resource constraint in steady state

$$\frac{c^*}{y^*} + \frac{g^*}{y^*} = 1 - \delta\frac{k^*}{y^*} = \frac{\rho + (1-\alpha)\delta}{\rho+\delta}$$

Since

$$\frac{g^*}{y^*} = \frac{\gamma}{1-\gamma}\frac{c^*}{y^*}$$

we then have simply

$$\frac{c^*}{y^*} = (1-\gamma)\left(\frac{\rho+(1-\alpha)\delta}{\rho+\delta}\right)$$

and

$$\frac{g^*}{y^*} = \gamma\left(\frac{\rho+(1-\alpha)\delta}{\rho+\delta}\right)$$

So finally

$$c^* = (1-\gamma)\left(\frac{\rho+(1-\alpha)\delta}{\rho+\delta}\right)\left(\frac{\alpha}{\rho+\delta}\right)^{\frac{\alpha}{1-\alpha}} A$$

and

$$g^* = \gamma\left(\frac{\rho+(1-\alpha)\delta}{\rho+\delta}\right)\left(\frac{\alpha}{\rho+\delta}\right)^{\frac{\alpha}{1-\alpha}} A$$

Hence all of c^*, g^*, k^*, y^* are directly proportional to A so that the ratios c^*/y^* and g^*/y^* are independent of A. The levels of k^* and y^* are independent of γ, which simply controls the relative preference for c and g. As γ increases g^* increases and c^* falls. Plugging in these parameter values we have $k^* = 26.8270$, $y^* = 2.6827$, $c^* = 1.5023$ and $g^* = 0.6438$ so that $c^*/y^* = 0.56$, $g^*/y^* = 0.24$ and $\delta k^*/y^* = 0.20$.

(d) Proceeding as in Lecture 5 slides 8–10 we have the log-linearized resource constraint

$$c^*\hat{c}_t + g^*\hat{g}_t + k^*\hat{k}_{t+1} = \frac{1}{\beta}k^*\hat{k}_t$$

and the log-linearized consumption Euler equation

$$\hat{c}_{t+1} = \hat{c}_t + \beta F^*_{kk}k^*\hat{k}_{t+1}$$

where, with this production function, $F^*_{kk} = \alpha(\alpha-1)(k^*)^{\alpha-2}A^{1-\alpha}$. We now also have the static optimality condition for c_t and g_t which, when log-linearized, is simply

$$\hat{c}_t = \hat{g}_t$$

We can then eliminate $\hat{g}_t$ from the resource constraint to write

$$(c^*+g^*)\hat{c}_t + k^*\hat{k}_{t+1} = \frac{1}{\beta}k^*\hat{k}_t$$

Plugging in our guesses and rearranging terms in this version of the resource constraint

$$\left[(c^*+g^*)\psi_{ck} + k^*\psi_{kk} - \frac{1}{\beta}k^*\right]\hat{k}_t = 0$$

This has to hold for any $\hat{k}_t$ hence we must have

$$(c^*+g^*)\psi_{ck} + k^*\psi_{kk} - \frac{1}{\beta}k^* = 0$$

or

$$\boxed{\psi_{ck} = \left(\frac{1}{\beta} - \psi_{kk}\right)\frac{k^*}{c^*+g^*}}$$

Likewise plugging in our guesses and rearranging terms in the consumption Euler equation

$$\left[\psi_{ck}\psi_{kk} - \psi_{ck} - \beta F^*_{kk} k^* \psi_{kk}\right] \hat{k}_t = 0$$

This has to hold for any $\hat{k}_t$ hence we must also have

$$\boxed{\psi_{ck}\psi_{kk} - \psi_{ck} - \beta F^*_{kk} k^* \psi_{kk} = 0}$$

Combining the two expressions in boxes gives a familiar looking quadratic in ψ_{kk}

$$\psi_{kk}^2 - \left(1 + \frac{1}{\beta} - \beta F^*_{kk}(c^* + g^*)\right)\psi_{kk} + \frac{1}{\beta} = 0$$

The roots of this quadratic *are* the eigenvalues of this dynamic system. There is one stable and one unstable root. Let ψ_{kk} denote the stable root. We can then recover ψ_{ck} from the first boxed equation. Finally, since $\hat{c}_t = \hat{g}_t$ it must also be true that $\psi_{ck} = \psi_{gk}$. With the given parameter values we get roots 1.0472 and 0.9645 hence $\psi_{kk} = 0.9645$. We then recover $\psi_{ck} = \psi_{gk} = 0.5672$.

These are exactly the same as we would get if there was no government consumption, $\gamma = 0$. To see this, notice that the sum $c^* + g^*$ is independent of the utility function $u(c, g)$ and hence independent of γ. Since the terms in c^*, g^* are the only way that γ can enter the system of log-linear equations and c^*, g^* always enter in the form $c^* + g^*$ the system of equations is independent of γ (which thus only affects steady state values, not the transitional dynamics around steady state).

(e) Using the expressions in part (b) above an increase in the level of productivity from A to $A' > A$ proportionately increases all of the steady state values from c^*, g^*, k^*, y^* to $c^{*\prime}, g^{*\prime}, k^{*\prime}, y^{*\prime}$ (say). Then relative to these new steady state levels the economy 'begins' with initial capital per worker $k_0 = k^* < k^{*\prime}$. On 'impact' the level of consumption immediately jumps to the new stable arm going through the new steady state. The level of government consumption also jumps so as to keep c_t/g_t constant. The level of output also jumps on impact because of the change in productivity. Capital does not jump on impact because it is predetermined. The level of total consumption $c_t + g_t$ jumps by less than the jump in output with the difference being saved. This increase in savings/investment is what allows the economy to build up a new higher level of capital in the long run. As the economy transitions to its new long run, c_t, g_t and y_t continue to rise with the new higher levels of capital. The ratio c_t/g_t always stays constant at $(1 - \gamma)/\gamma$ regardless of the level of productivity.